

The value of a zero-sum stochastic differential game involving impulse control

Parsiad Azimzadeh*

September 30, 2016

Abstract

We study a finite-horizon zero-sum stochastic differential game in which one player plays an impulse control and their opponent plays a stochastic control. We establish that the game admits a value, and in turn, the existence and uniqueness of viscosity solutions to an associated Hamilton-Jacobi-Bellman-Isaacs equation. We perform our analysis with minimal regularity so that the value functions are not a priori continuous. This poses some interesting challenges.

1 Introduction

The theory of *stochastic differential games* (SDGs) can be traced back to the introduction of deterministic differential games (DDGs) by Isaacs [23]. Using the notion of Elliot-Kalton strategies [17], Evans and Souganidis considered DDGs using viscosity theory [18]. This was followed by the pioneering work of Fleming and Souganidis, who also used viscosity theory to consider SDGs [19]. Since then, SDGs have been studied under different settings (e.g., asymmetric information) and using various tools (e.g., backwards stochastic differential equations, path dependent partial differential equations, etc.). We list a few such works here: [20, 21, 8, 10, 22, 6, 32, 29, 16].

The references listed above are mainly concerned with SDGs under stochastic controls. We consider instead an *impulse control* problem, in which the actions of a player affect the system in an “instantaneous” manner. We are aware of only a few works [36, 13] that study SDGs with impulse control (along with a related study on switching controls in [33]). This is in spite of the fact that impulse control problems have enjoyed a resurgence (see, e.g., [2, 11]) due to a demand for more realistic financial models (e.g., fixed transaction costs and liquidity risk) [27, 5, 31, 12] and their link to backwards stochastic differential equations [26].

In our game, two players compete on a finite horizon $[t, T]$ by influencing a stochastic process, denoted X . The “sup-player” aims to maximize a particular function, while the “inf-player” aims to minimize it.

The sup-player exerts their control by choosing impulse times $\tau_1 \leq \tau_2 \leq \dots$ and impulse controls z_1, z_2, \dots , denoted $a := (\tau_j, z_j)_j$ for brevity. The inf-player exerts their control by

*David R. Cheriton School of Computer Science, University of Waterloo, Waterloo ON, Canada N2L 3G1
pazimzad@uwaterloo.ca.

choosing a process $(b_t)_t$. Letting $(W_t)_t$ denote a standard Brownian motion, between impulse times, X follows the stochastic differential equation (SDE)

$$dX_s = \mu(X_s, b_s)ds + \sigma(X_s, b_s)dW_s.$$

At an impulse time τ_j , the process changes instantaneously as a function of the corresponding impulse control z_j :

$$X_{\tau_j} = X_{\tau_j-} + \Gamma(\tau_j, z_j)$$

where $t-$ is shorthand for a limit from the left.

As is usually the case in SDGs, players play not controls (a, b) but rather strategies (α, β) . Given a functional $J := J(t, x; a, b)$ whose first two arguments describe the initial time and state of the process X , the upper and lower values of the game are

$$\inf_{\beta} \sup_a J(t, x; a, \beta(a)) \text{ and } \sup_{\alpha} \inf_b J(t, x; \alpha(b), b).$$

When the upper and lower values coincide, we say that the game admits a *value*.

To establish that the game admits a value, we apply the viscosity theory to an associated Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation. In particular, we establish three results, for which we give here high-level overviews:

1. A *dynamic programming principle* (DPP) relates the upper (resp. lower) value of the game to itself at a future stopping time θ . As is usually the case in SDGs, θ cannot be any arbitrary stopping time: it can only take at most countably many values (i.e., $\text{range}(\theta)$ is countable).
2. A *dynamic programming equation* (DPE) establishes that the upper (resp. lower) value is a viscosity subsolution (resp. supersolution) of the HJBI.
3. A *comparison principle* guarantees that solutions to the HJBI are unique.

The above program is used to establish that the upper value is no less than the lower value. The reverse inequality is a consequence of the delay we impose on the strategies.

We momentarily digress to point out that previous works in impulse control often ignore the well-posedness of the underlying stochastic differential equation (SDE), which can blow up in finite time due to impulses (e.g., consider the time-horizon $[0, 1]$ with $\Gamma(t, z) := z$ and deterministic impulses of size $z_j := 1/j$ occurring at each time $\tau_j := 1 - 2^{-j}$). The technique employed in our work to guarantee well-posedness can, with little hassle, be used to strengthen the results of [36, 13].

In order to remain as general as possible, we avoid imposing unnecessary regularity on J . A by-product of this is that the upper and lower value functions are not known to be a priori continuous, which leads to some interesting problems in the derivation of the DPE, discussed below.

In the “single player” setting, one derives a DPE in the viscosity sense by fixing a point (t, x) and a test function φ making contact with the value function at that point. One then finds an open set $\mathcal{N}_{2h} \ni (t, x)$ where $\mathcal{N}_h := (t - h, t + h) \times \{y : |x - y| < h\}$ and inside which certain desirable local properties are satisfied. By considering the “début-type” stopping time

$$\theta := \inf \{s > t : (s, X_s) \notin \mathcal{N}_h\},$$

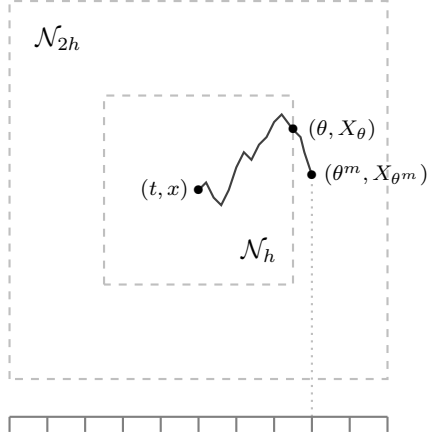


Figure 1.1: Approximating θ by θ^m

one can invoke the DPP while ensuring that $X_\theta \in \text{cl } \mathcal{N}_h$, so that the desirable local properties satisfied on \mathcal{N}_{2h} can be employed. However, we are unable to use such a stopping time as its range is uncountable!

The problem described above can sometimes be circumvented by using the a priori uniform continuity of the value functions (e.g., Lipschitz continuous in x , Hölder continuous in t) which is obtained, roughly speaking, by imposing sufficient regularity on J . This approach is used in [13].

However, we wish to perform our analyses devoid of a priori continuity. Instead, we approximate θ by a sequence of stopping times $(\theta^m)_m$ given by

$$\theta^m := \min(\{kT/m : k \geq 1\} \cap [\theta, T]).$$

Note, in particular, that the range of θ^m is finite (see Figure 1.1). Letting $A_m := \{(\theta^m, X_{\theta^m}) \in \mathcal{N}_{2h}\}$, we retrieve $\mathbf{1}_{A_m} \rightarrow 1$ almost surely from the continuity of sample paths of X (between impulses). By considering the problem on the set A_m and its complement, we use an argument involving compact test functions to show that the contributions from the complement are zero (or at least, as close to zero as we desire), finally retrieving the desired DPE.

2 Framework and statement of results

Fix $T \in [0, \infty)$ and an \mathbb{R}^{d_W} -valued standard Brownian motion $(W_t)_{t \geq 0}$ on the canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\Omega_{t,T}$ be the set of continuous functions from $[t, T]$ to \mathbb{R}^d starting at zero and $\mathbb{P}_{t,T}$ its associated Wiener measure (i.e., $\Omega = \Omega_{0,T}$ and $\mathbb{P} = \mathbb{P}_{0,T}$). We denote by $\mathcal{F}_{t,s}$ the σ -algebra generated by $(W_u - W_t)_{u \in [t,s]}$ and augmented by all \mathbb{P} null sets. We omit the subscripts on Ω and \mathbb{P} (and the associated expectation \mathbb{E}) whenever it is unambiguous to do so.

Definition 2.1 (Controls). A $[t, T]$ impulse control is a tuple $a := (\tau_j, z_j)_j$ where $\tau_1 \leq \tau_2 \leq \dots$ is a nondecreasing sequence of $(\mathcal{F}_{t,s})_{s \in [t,T]}$ -stopping times, each z_j is an $\mathcal{F}_{t, \tau_j \wedge T}$ -measurable random variable taking values in $Z \subset \mathbb{R}^{d_Z}$, and

(C0) $\mathbb{E}[\#a] < \infty$ where $\#a := \max\{j \geq 1 : \tau_j \leq T\}$;

(C1) each τ_j takes values in $\mathbb{Q}_t \cup \{+\infty\}$ where $\mathbb{Q}_t := ([t, T] \cap \mathbb{Q}) \cup \{T\}$.

The set of all such controls is denoted $\mathcal{A}(t)$.

A $[t, T]$ stochastic control is an $(\mathcal{F}_{t,s})_{s \in [t, T]}$ -progressively measurable process $b := (b_s)_{s \in [t, T]}$ taking values in $B \subset \mathbb{R}^{d_B}$. The set of all such controls is denoted $\mathcal{B}(t)$.

Remark. (C0) states that on average, only finitely many impulses occur (see also [13, Definition 2.5]). (C1) states that impulses occur only at rational times (or at T). (C1) is, at least intuitively, not restrictive since any stopping time can be approximated (from above) by a sequence of rational stopping times.

We can actually replace \mathbb{Q} in (C1) with any dense countable subset of \mathbb{R} , though we avoid this generality so as to not overburden the notation.

Given controls a and b as above, the relevant SDE (with impulses) is

$$X_s = x + \int_t^s \mu(X_u, b_u) du + \int_t^s \sigma(X_u, b_u) dW_u + \sum_{\tau_j \leq s} \Gamma(\tau_j, z_j) \text{ for } s \in [t, T]. \quad (2.1)$$

If it exists and is unique, we use $X^{t,x;a,b}$ to denote a solution (see Definition 2.2) to (2.1). The gain (resp. cost) functional for the sup (resp. inf)-player is given by

$$J(t, x; a, b) := \mathbb{E} \left[\int_t^T f(s, X_s, b_s) ds + \sum_{\tau_j} K(\tau_j, z_j) + g(X_T) \right]$$

where it is understood that $X := X^{t,x;a,b}$. It is convenient at this point to also define the intervention operator \mathcal{M} , which (roughly speaking) describes the value of the game immediately after an optimal impulse:

$$\mathcal{M}u(t, x) := \sup_{z \in Z} \{u(t, x + \Gamma(t, z)) + K(t, z)\}. \quad (2.2)$$

We are now ready to introduce admissible controls and strategies.

Definition 2.2 (Admissible impulse control). A $[t, T]$ impulse control $a \in \mathcal{A}(t)$ is admissible at $x \in \mathbb{R}^d$ if for all $b \in \mathcal{B}(t)$, a solution of (2.1) exists and is unique. The set of all such controls is denoted $\mathcal{A}(t, x)$.

By a *solution* $X := X^{t,x;a,b}$, we mean that X is $(\mathcal{F}_{t,s})_{s \in [t, T]}$ -adapted, has càdlàg paths, is in $\mathbb{L}^2(\Omega_{t,T} \times [t, T])$, and satisfies (2.1). Uniqueness is determined up to indistinguishability.

Let $\bar{t}, t \in [0, T]$ with $\bar{t} \leq t$. Given $\omega \in \Omega_{\bar{t}, T}$, we define (ω_1, ω_2) by

$$\begin{aligned} \omega_1 &:= \omega|_{[\bar{t}, t]} \\ \text{and } \omega_2 &:= (\omega - \omega(t))|_{[t, T]}, \end{aligned}$$

identifying $\Omega_{\bar{t}, T}$ with $\Omega_{\bar{t}, t} \times \Omega_{t, T}$ along with $\mathbb{P}_{\bar{t}, T} = \mathbb{P}_{\bar{t}, t} \otimes \mathbb{P}_{t, T}$. We notice that

- for $b \in \mathcal{B}(\bar{t})$, the control $b|_{[t, T]}(\omega_1)$ defined by $(b|_{[t, T]}(\omega_1))(\omega_2)_s := b(\omega)_s$ is a member of $\mathcal{B}(t)$ for $\mathbb{P}_{\bar{t}, t}$ -almost all ω_1 ;

- for $a := (\tau_j, z_j)_j \in \mathcal{A}(\bar{t})$, the control $a|_{(t,T]}(\omega_1) := (\hat{\tau}_j(\omega_1), \hat{z}_j(\omega_1))_j$ defined by $(\hat{\tau}_j(\omega_1))(\omega_2) := \tau_{\ell(\omega)+j}(\omega)$, $(\hat{z}_j(\omega_1))(\omega_2) := z_{\ell(\omega)+j}(\omega)$, and $\ell(\omega) := \min\{j \geq 1: \tau_j(\omega) > t\} - 1$ is a member of $\mathcal{A}(t)$ for $\mathbb{P}_{\bar{t},t}$ -almost all ω_1 .

Before we give the next definition, note that each $(\mathcal{F}_{t,s})_{s \in [t,T]}$ -stopping time τ can be identified with a $\{0,1\}$ -valued, adapted, nondecreasing, right-continuous process $G(\tau)$ by taking $G(\tau)_s := \mathbf{1}_{\{\tau \leq s\}}$ [35]. For two such stopping times τ, τ' and $s \in [t, T]$, this gives an obvious meaning to the statement “ $\tau = \tau'$ on $[t, s]$.”

Definition 2.3 (Control identification). For $a := (\tau_j, z_j)_j$ and $a' := (\tau'_j, z'_j)_j$ in $\mathcal{A}(t)$ and $s \in [t, T]$, we write $a \equiv a'$ on $[t, s]$ if

$$\mathbb{P}(\tau_j = \tau'_j \text{ on } [t, s] \text{ and } (z_j - z'_j)\mathbf{1}_{\{\tau_j \leq s\}} = 0 \text{ for all } j) = 1.$$

For $b, b' \in \mathcal{B}(t)$ and $s \in [t, T]$, we write $b \equiv b'$ on $[t, s]$ if

$$\mathbb{P}(b = b' \text{ almost everywhere on } [t, s]) = 1.$$

Definition 2.4 (Strategies). $\alpha : \mathcal{B}(t) \rightarrow \mathcal{A}(t)$ is an impulse strategy if it is

- (i) nonanticipative: for any $b, b' \in \mathcal{B}(t)$ and $s \in [t, T]$, if $b \equiv b'$ on $[t, s]$, then $\alpha(b) \equiv \alpha(b')$ on $[t, s]$;
- (ii) delayed: there is a partition $t = t_0 < t_1 < \dots < t_m = T$ such that for all $b, b' \in \mathcal{B}(t)$ and $i < m$, if $b \equiv b'$ on $[t, t_i]$, $\alpha(b) \equiv \alpha(b')$ on $[t, t_{i+1}]$.

The set of all such strategies is denoted $\mathcal{A}(t)$. Moreover, denote by $\mathcal{A}(t, x)$ the set of all impulse strategies $\alpha \in \mathcal{A}(t)$ with $\text{range}(\alpha) \subset \mathcal{A}(t, x)$.

$\beta : \mathcal{A}(t) \rightarrow \mathcal{B}(t)$ is a stochastic strategy if it is

- (i) nonanticipative;
- (ii) delayed;
- (iii) an r -strategy: for every $\bar{t}, t \in [0, T]$ with $\bar{t} < t$ and $a \in \mathcal{A}(\bar{t})$, the process $\beta(a|_{(t,T]})$ given by

$$\beta(a|_{(t,T]})(\omega)_r := \beta(a|_{(t,T]}(\omega_1))(\omega_2)_r \text{ where } r \in [t, T]$$

is $(\mathcal{F}_{\bar{t},s})_{s \in [\bar{t}, T]}$ -progressively measurable.

The set of all such strategies is denoted $\mathcal{B}(t)$.

Nonanticipativity disallows a player from using future information from their opponent's control. Strategies with delay (see also [10, 9]) are used to ensure that the upper value of the game is no less than the lower value (see §7). *r-strategies* (for restricted) were introduced in [19, Definition 1.7] to overcome certain measurability issues.

We introduce below the concept of a *nonanticipative family of stopping times* that formalizes the intuitive notion that the decision to stop should not depend on future information from the controls. To the best of our knowledge, these have not been introduced previously, and are used to ensure that the strategies constructed in the proof of the DPP are nonanticipative.

Definition 2.5. Let $t \in [0, T]$ and $\mathcal{C}(t)$ be a subset of $\mathcal{A}(t) \times \mathcal{B}(t)$. $\{\theta^{a,b}\}_{(a,b) \in \mathcal{C}(t)}$ is a nonanticipative family of $(\mathcal{F}_{t,s})_{s \in [t, T]}$ -stopping times if

- (i) $\theta^{a,b}$ is an $(\mathcal{F}_{t,s})_{s \in [t,T]}$ -stopping time for each $(a,b) \in \mathcal{C}(t)$;
- (ii) for each $s \in [t,T]$ and $(a,b), (a',b') \in \mathcal{C}(t)$, if $a \equiv a'$ and $b \equiv b'$ on $[t,s]$, then $\mathbb{P}(\theta^{a,b} = \theta^{a',b'} \text{ on } [t,s]) = 1$.

Remark. Though we abstain from doing so for ease of exposition, we can extend our results to “strongly” nonanticipative strategies as introduced in [7, Definition 2.3] (of course, this requires extending, in the obvious way, the definition above to “strongly” nonanticipative families of stopping times).

We are now ready to introduce the upper and lower values of the game:

$$v^+(t, x) := \inf_{\beta \in \mathcal{B}(t)} \sup_{a \in \mathcal{A}(t, x)} J(t, x; a, \beta(a)) \text{ and } v^-(t, x) := \sup_{\alpha \in \mathcal{A}(t, x)} \inf_{b \in \mathcal{B}(t)} J(t, x; \alpha(b), b).$$

The game is said to admit a value if $v^+ = v^-$ pointwise.

We gather some assumptions below, which are understood to hold throughout the text. These assumptions are discussed further at the end of this section.

Assumption 2.6. (i) Z is closed and nonempty and B is compact and nonempty;

- (ii) $\mu : \mathbb{R}^d \times B \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times B \rightarrow \mathbb{R}^{d \times dw}$ are continuous and Lipschitz in x (uniformly in b):

$$|\mu(x, b) - \mu(y, b)| + |\sigma(x, b) - \sigma(y, b)| \leq \text{const.} |x - y|;$$

- (iii) $\Gamma : [0, T] \times Z \rightarrow \mathbb{R}^d$ is continuous.

Assumption 2.7. (i) the functions $f : [0, T] \times \mathbb{R}^d \times B \rightarrow \mathbb{R}$, $K : [0, T] \times Z \rightarrow \mathbb{R}$, and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ are continuous;

- (ii) f and g are bounded and Lipschitz in x (uniformly in t and b):

$$|f(t, x, b) - f(t, y, b)| + |g(x) - g(y)| \leq \text{const.} |x - y|;$$

- (iii) $z \mapsto K(t, z) \in \omega(1)$ as $|z| \rightarrow \infty$ (uniformly in t)¹ and there exists a positive constant K_0 such that $K \leq -K_0$ pointwise.

Assumption 2.8. (i) for each $t \in [0, T]$ and $z_1, z_2 \in Z$, there exists $z \in Z$ such that $\Gamma(t, z) = \Gamma(t, z_1) + \Gamma(t, z_2)$ and $K(t, z) \geq K(t, z_1) + K(t, z_2)$.

- (ii) for each sequence $(t_n, x_n)_n$ taking values in $[0, T] \times \mathbb{R}^d$ and converging to some (T, x) , $\liminf_{n \rightarrow \infty} v^-(t_n, x_n) \geq g(x)$ and if $v^+(t_n, x_n) > \mathcal{M}v^+(t_n, x_n)$ for all n , $\limsup_{n \rightarrow \infty} v^+(t_n, x_n) \leq g(x)$.

For a locally bounded above (resp. below) function u from some metric space Y to \mathbb{R} , we use u^* (resp. u_*) to denote the upper (resp. lower) semicontinuous envelope of u . Recall that the upper (resp. lower) semicontinuous envelope of a function u is the smallest (resp. largest) upper (resp. lower) semicontinuous function w such that $w \geq u$ (resp. $w \leq u$). Unless otherwise mentioned, Y is taken to be $[0, T] \times \mathbb{R}^d$.

¹Here, ω is the Bachmann–Landau symbol. Precisely, we mean that for each $k > 0$, there exists an $r > 0$ such that for all $(t, z) \in [0, T] \times Z$ with $|z| > r$, $|K(t, z)| \geq k$.

We are now in a position to state the main results of this work. For brevity, let $\mathcal{O} := [0, T) \times \mathbb{R}^d$, $\partial^+ \mathcal{O} := \{T\} \times \mathbb{R}^d$ denote the parabolic boundary of \mathcal{O} , and

$$J_0(t, x; a, b; \theta) := \int_t^\theta f(s, X_s, b_s) ds + \sum_{\tau_j \leq \theta} K(\tau_j, z_j).$$

We begin with the DPP, whose proof is given in §4. In the statement below, for each integer $q \geq 1$ and Borel set $Q \subset \mathbb{R}^{dz}$, $\mathcal{A}^{q, Q}(t)$ is the set of all controls $a := (\tau_j, z_j)_j \in \mathcal{A}(t)$ such that

$$\mathbb{P}(S(a)) = 1 \text{ where } S(a) := (\cap_{j \geq q} \{\tau_j > T\}) \cap (\cap_j \{\tau_j > T\} \cup \{z_j \in Q\}).$$

Intuitively, $\mathcal{A}^{q, Q}(t)$ is the set of controls with less than q impulses and with each impulse contained in the set Q . Similarly, we denote by $\mathcal{A}^{q, Q}(t)$ the set of all strategies $\alpha \in \mathcal{A}(t)$ with $\text{range}(\alpha) \subset \mathcal{A}^{q, Q}(t)$.

Theorem 2.9 (DPP). *There exists an integer $q \geq 1$ and a compact set $Q \subset \mathbb{R}^{dz}$ such that for each $(t, x) \in \mathcal{O}$ and each nonanticipative family of \mathbb{Q}_t -valued $(\mathcal{F}_{t,s})_{s \in [t, T]}$ -stopping times $\{\theta^{a,b}\}_{(a,b) \in \mathcal{A}^{q, Q}(t) \times \mathcal{B}(t)}$,*

$$\begin{aligned} v^+(t, x) &\leq \inf_{\beta \in \mathcal{B}(t)} \sup_{a \in \mathcal{A}^{q, Q}(t)} \mathbb{E} [J_0(t, x; a, \beta(a); \theta) + (v^+)^*(\theta, X_\theta)] \\ (\text{resp. } v^-(t, x) &\geq \sup_{\alpha \in \mathcal{A}^{q, Q}(t)} \inf_{b \in \mathcal{B}(t)} \mathbb{E} [J_0(t, x; \alpha(b), b; \theta) + (v^-)_*(\theta, X_\theta)]) \end{aligned}$$

where it is understood that $X := X^{t, x; a, \beta(a)}$ and $\theta := \theta^{a, \beta(a)}$ (resp. $X := X^{t, x; \alpha(b), b}$ and $\theta := \theta^{\alpha(b), b}$).

The HJBI associated with the game is a quasi-variational inequality:

$$0 = F(\cdot, u, Du(\cdot), D^2u(\cdot)) := \begin{cases} \min\{-\inf_{b \in B}\{(\partial_t + \mathbb{L}^b)u + f^b\}, u - \mathcal{M}u\} & \text{on } \mathcal{O} \\ \min\{u - g, u - \mathcal{M}u\} & \text{on } \partial^+ \mathcal{O} \end{cases} \quad (2.3)$$

where $f^b(t, x) := f(t, x, b)$, \mathcal{M} is defined by (2.2), and

$$\mathbb{L}^b u(t, x) := \frac{1}{2} \text{trace}(\sigma(x, b) \sigma^\top(x, b) D_x^2 u(t, x)) + \langle \mu(x, b), D_x u(t, x) \rangle.$$

We point out that due to the operator \mathcal{M} , (2.3) is nonlocal in its use of u . No second time derivatives appear and so we use D^2 and D_x^2 interchangeably, while D is interpreted to mean either (∂_t, D_x) or D_x , depending on context. Since the above is only formal, we need to ascribe meaning to the notion of a “solution” to (2.3):

Definition 2.10 (Viscosity solution). A locally bounded above (resp. below) function $u : \text{cl } \mathcal{O} \rightarrow \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of (2.3) if, letting $w := u^*$ (resp. $w := u_*$), for all $(t, x, \varphi) \in \text{cl } \mathcal{O} \times C^{1,2}(\text{cl } \mathcal{O})$ such that $(w - \varphi)(t, x)$ is a local maximum (resp. minimum) of $w - \varphi$,

$$F(t, x, w, D\varphi(t, x), D^2\varphi(t, x)) \leq 0 \text{ (resp. } \geq 0).$$

u is said to be a viscosity solution of (2.3) whenever it is simultaneously a supersolution and subsolution of (2.3).

The definition above can be applied to any second-order (nonlocal in the zero-th order term) operator F defined on $\text{cl } \mathcal{O}$, not just (2.3). We mention that the notion of viscosity solution above is sometimes referred to as “strong” (see, e.g., [24, Definition 2.2]) due to the appearance of the boundary conditions. We hereafter drop the term “viscosity” in our discussions, since it is the main solution concept in this work.

We can now state the relationship between the upper and lower value functions of the game and the HJBI; namely that the HJBI is the DPE associated with the game. A proof of this fact is given in §5.

Theorem 2.11 (DPE). v^+ (resp. v^-) is a bounded subsolution (resp. supersolution) of (2.3).

We also establish a comparison principle for the HJBI, with proof given in §6.

Theorem 2.12 (Comparison principle). If u is a bounded subsolution and w is a bounded supersolution of (2.3), $u^* \leq w_*$ pointwise.

We use the results of Theorems 2.11 and 2.12 to establish that the game admits a value, with proof given in §7.

Theorem 2.13 (Value). $v^+ = v^-$ pointwise.

A consequence of the above results is the following existence and uniqueness claim:

Corollary 2.14 (HJBI existence and uniqueness). v^\pm is a continuous solution of (2.3), unique among all bounded solutions.

To conclude this section, we discuss the significance of the assumptions listed above, as promised.

- Assumption 2.6 (i) and (ii) ensure the existence and uniqueness of solutions to (2.1) when a is a control with no impulses (implying that $\mathcal{A}(t, x)$ is nonempty). (iii) is needed to extend this result to nontrivial impulse controls.
- Assumption 2.7 is used to establish the regularity and finitude of the cost functional (see §3).
- Assumption 2.8 (i) ensures that multiple impulses occurring at the same time are suboptimal, while (ii) ensures that v^\pm satisfies the boundary conditions of the HJBI (see also [31, Assumption (E3)]). Example 5.2 presents a common situation in which (ii) is satisfied.

3 Regularity

We first make clear the default norms used in this work:

Notation. Let $\langle x, y \rangle$ be the Euclidean inner product, $|x| := \sqrt{\langle x, x \rangle}$, and $B(x; r)$ be the ball (in the associated metric) of radius $r > 0$ centred at x . We use $|A|$ to denote the spectral radius of a matrix A .

Before we begin, we make a few observations. Let $t \in [0, T]$ and $a := (\tau_j, z_j)_j \in \mathcal{A}(t)$. The set

$$\{\#a < \infty\} = \cup_{n \geq 1} \cap_{j \geq n} \{\tau_j > T\}$$

has full measure by (C0), since otherwise $\mathbb{E}[\#a] = +\infty$. We can therefore treat sums \sum_{τ_j} (under expectations) as having finitely many terms. Moreover, any of the Lipschitz functions defined in §2 are necessarily of linear growth. For example,

$$|\mu(x, b)| \leq |\mu(x, b) - \mu(0, b)| + |\mu(0, b)| \leq \text{const.} |x| + \sup_{b \in B} |\mu(0, b)|,$$

and the desired growth follows from the continuity of μ and compactness of B .

Lemma 3.1. *v^+ and v^- are bounded.*

Proof. Let $c := T\|f\|_\infty + \|g\|_\infty$ and $(t, x) \in \text{cl } \mathcal{O}$. Since the sup-player is free to play a control $\hat{a} \in \mathcal{A}(t, x)$ with no impulses,

$$v^+(t, x) \geq \inf_{\beta \in \mathcal{B}(t)} \mathbb{E} \left[\int_t^T f(s, X_s, \beta(\hat{a})_s) ds + g(X_T) \right] \geq -c \quad (3.1)$$

where it is understood that $X := X^{t, x; \hat{a}, \beta(\hat{a})}$. Moreover, since $K \leq 0$,

$$v^+(t, x) \leq \inf_{\beta \in \mathcal{B}(t)} \sup_{a \in \mathcal{A}(t, x)} \mathbb{E} \left[\int_t^T f(s, X_s, \beta(a)_s) ds + g(X_T) \right] \leq c \quad (3.2)$$

where it is understood that $X := X^{t, x; a, \beta(a)}$. The same arguments hold for the lower value function v^- . \blacksquare

We defined the notion of an admissible control based on the existence and uniqueness of a strong solution to the SDE. It is not possible for us to perform our analyses on these control sets without running into trouble. The next two results alleviate this by a simple characterization of $\mathcal{A}^{q, Q}(t) \times \mathcal{B}(t)$ and the value functions.

Lemma 3.2. *Let $q \geq 1$ be an integer, $Q \subset \mathbb{R}^{d_z}$ be a compact set, $(t, x) \in [0, T]$, and $(a, b) \in \mathcal{A}^{q, Q}(t) \times \mathcal{B}(t)$. Then, there exists a unique solution to (2.1).*

Proof. Let X and \hat{X} be two solutions (with the same initial condition x). Letting $\delta X_s := X_s - \hat{X}_s$, we arrive at $\mathbb{E}[\delta X_s] = 0$ for all $s \in [t, T]$ so that

$$\mathbb{P}(\delta X_s = 0 \text{ for all } s \in \mathbb{Q} \cap [t, T]) = 1.$$

Since paths are right continuous,

$$\mathbb{P}(\delta X_s = 0 \text{ for all } s \in [t, T]) = 1$$

and hence uniqueness is proved.

Let $Y_s^{(0)} := x + \sum_{\tau_j \leq s} \Gamma(\tau_j, z_j)$ and define inductively

$$Y_s^{(k+1)} := x + \int_t^s \mu(Y_u^{(k)}, b_u) du + \int_t^s \sigma(Y_u^{(k)}, b_u) dW_u + \sum_{\tau_j \leq s} \Gamma(\tau_j, z_j).$$

Before we continue, we should check that $Y^{(k+1)} \in \mathbb{L}^2(\Omega_{t, T} \times [t, T])$ whenever $Y^{(k)} \in \mathbb{L}^2(\Omega_{t, T} \times [t, T])$. Using the linear growth of μ and σ , we can show that

$$\mathbb{E} \left[\int_t^T \left| \int_t^s \mu(Y_u^{(k)}, b_u) du \right|^2 ds \right] + \mathbb{E} \left[\int_t^T \left| \int_t^s \sigma(Y_u^{(k)}, b_u) du \right|^2 ds \right] < \infty.$$

Moreover,

$$\mathbb{E} \left[\int_t^T \left| \sum_{\tau_j \leq s} \Gamma(\tau_j, z_j) \right|^2 ds \right] \leq \text{const.} \sup_{(t,z) \in [0,T] \times Q} |\Gamma(t,z)|^2 < \infty$$

where const. can depend on q . The above also shows $Y^{(0)} \in \mathbb{L}^2(\Omega_{t,T} \times [t,T])$.

The remainder of the proof is identical to [28, Theorem 5.2.1] with the exception that Doob's inequality for càdlàg martingales [30, Theorem 1.7] should be used instead of [28, Theorem 3.2.4]. \blacksquare

Lemma 3.3. *There exists an integer $q \geq 1$ and a compact set $Q \subset \mathbb{R}^{dz}$ such that for all $(t,x) \in \text{cl } \mathcal{O}$,*

$$v^+(t,x) := \inf_{\beta \in \mathcal{B}(t)} \sup_{a \in \mathcal{A}^{q,Q}(t)} J(t,x;a,\beta(a))$$

$$\text{and } v^-(t,x) := \sup_{\alpha \in \mathcal{A}^{q,Q}(t)} \inf_{b \in \mathcal{B}(t)} J(t,x;\alpha(b),b).$$

Proof. Let $c := T\|f\|_\infty + \|g\|_\infty$. By Assumption 2.7 (iii), there is an $r > 0$ such that for $|z| > r$, $K(t,z) \leq -2c$ for all t . Similarly, since $K \leq -K_0 < 0$, there exists an integer $q \geq 1$ such that $-qK_0 \leq -2c$. \blacksquare

For the remainder of this section, fix an integer $q \geq 1$ and a compact set $Q \subset \mathbb{R}^{dz}$. When we restrict our attention to controls in $\{\mathcal{A}^{q,Q}(t)\}_{t \in [0,T]}$, J is bounded:

Lemma 3.4. $|J(t,x;a,b)| \leq \text{const.}$ for all $(t,x) \in \text{cl } \mathcal{O}$ and $(a,b) \in \mathcal{A}^{q,Q}(t) \times \mathcal{B}(t)$.

Proof. Let $c := T\|f\|_\infty + \|g\|_\infty$ and $K_1 := \sup_{[0,T] \times Q} |K(t,z)|$. Then,

$$J(t,x;a,b) \geq -c - \mathbb{E}[\#a] K_1 > -c - qK_1.$$

The upper bound follows the same line of reasoning as (3.2). \blacksquare

In the sequel, we need to construct a countable Borelian partition on which players can make “ ϵ -optimal” choices. To do so, we require J to exhibit uniform continuity in space, independent of the other arguments:

Lemma 3.5. $|J(t,x;a,b) - J(t,\hat{x};a,b)| \leq \text{const.} |x - \hat{x}|$ for all $t \in [0,T]$ and $(a,b) \in \mathcal{A}^{q,Q}(t) \times \mathcal{B}(t)$.

Proof. Let $X := X^{t,x;a,b}$ and $\hat{X} := X^{t,\hat{x};a,b}$. Further letting $\delta x := x - \hat{x}$, $\delta X_s := X_s - \hat{X}_s$, $\delta \mu_s := \mu(X_s, b_s) - \mu(\hat{X}_s, b_s)$, and $\delta \sigma_s := \sigma(X_s, b_s) - \sigma(\hat{X}_s, b_s)$, we can use Hölder's inequality, Itô isometry, the Lipschitz continuity of μ and σ , and the Fubini-Tonelli theorem to get

$$\begin{aligned} \mathbb{E} [|\delta X_s|^2] &\leq \text{const.} \mathbb{E} \left[|\delta x|^2 + \left| \int_t^s \delta \mu_u du \right|^2 + \left| \int_t^s \delta \sigma_u dW_u \right|^2 \right] \\ &\leq \text{const.} \mathbb{E} \left[|\delta x|^2 + T \int_t^s |\delta \mu_u|^2 du + \int_t^s |\delta \sigma_u|^2 du \right] \\ &\leq \text{const.} \left(|\delta x|^2 + \int_t^s \mathbb{E} [|\delta X_u|^2] du \right) \end{aligned}$$

for $s \in [t, T]$. Now, an application of Grönwall's lemma followed by Jensen's inequality (for concave functions) yields

$$\mathbb{E}[|\delta X_s|] \leq \text{const.} |\delta x| \text{ for } s \in [t, T]. \quad (3.3)$$

Note, in particular, that const. depends only on quantities such as T and the Lipschitz constants of μ and σ . Moreover, defining $\delta f_s := f(s, X_s, b_s) - f(s, \hat{X}_s, b_s)$ and $\delta g_T := g(X_T) - g(\hat{X}_T)$,

$$|J(t, x; a, b) - J(t, \hat{x}; a, b)| \leq \mathbb{E} \left[\int_t^T |\delta f_s| ds + |\delta g_T| \right]$$

and the desired result follows from applying the Lipschitz continuity of f and g and (3.3) to the inequality above. \blacksquare

While the above implies the Lipschitz continuity of the value functions in x for each fixed t , it does not imply continuity on $[0, T] \times \mathbb{R}^d$! Such continuity requires additional conditions on K (see, e.g., [13, (2.6)]), which we avoid.

4 Dynamic programming principle

We prove, in this section, Theorem 2.9. Only the first inequality in the theorem statement is proved below, the other being similar.

Proof of Theorem 2.9. Let $\epsilon > 0$ and $\varphi \geq v^+$ be a continuous function bounded from above by a constant. For each $(s, y) \in \mathcal{U} := [t, T] \times \mathbb{R}^d$, there exists $\beta^{s,y} \in \mathcal{B}(s)$ such that

$$\varphi(s, y) \geq v^+(s, y) \geq J(s, y; a, \beta^{s,y}(a)) - \epsilon/6 \text{ for all } a \in \mathcal{A}^{q,Q}(s).$$

Using the continuity of J established in Lemma 3.5 and that of φ , we can find a family of positive constants $\{r^{s,y}\}_{(s,y) \in \mathcal{U}}$ such that

$$\begin{aligned} J(s, y; a, b) &\geq J(s, x'; a, b) - \epsilon/6 \text{ and } \varphi(s, x') \geq \varphi(s, y) - \epsilon/6 \\ &\text{for } (s, x') \in [t, T] \times B(y; r^{s,y}) \text{ and } (a, b) \in \mathcal{A}^{q,Q}(s) \times \mathcal{B}(s). \end{aligned}$$

Let $\{t_i\}_{i \geq 1} := \mathbb{Q}_t$. Since $\{B(y; r^{s,y})\}_{y \in \mathbb{R}^d}$ is a cover of \mathbb{R}^d by open balls (for each $s \in [t, T]$), for each i , Lindelöf's lemma yields the countable subcover $\{B(x_{i,j}; r_{i,j})\}_{j \geq 1}$ of \mathbb{R}^d where $r_{i,j} := r^{t_i, x_{i,j}}$ for brevity. To turn this subcover into a Borelian partition, take $C_{i,0} := \emptyset$ and

$$A_{i,j} := (\{t_i\} \times B(x_{i,j}; r_{i,j})) \setminus C_{i,j-1} \text{ where } C_{i,j} := A_{i,1} \cup \dots \cup A_{i,j} \text{ for } j \geq 1.$$

The family $\{A_{i,j}\}_{j \geq 1}$ is disjoint by construction. Denote by $A^n := \cup_{i \leq n} \cup_{j \leq n} A_{i,j}$. Letting $\beta_{i,j} := \beta^{t_i, x_{i,j}}$, we have

$$\begin{aligned} \varphi(t_i, x') &\geq \varphi(t_i, x_{i,j}) - \epsilon/6 \\ &\geq J(t_i, x_{i,j}; a, \beta_{i,j}(a)) - \epsilon/3 \\ &\geq J(t_i, x'; a, \beta_{i,j}(a)) - \epsilon/2 \quad \text{for } (t_i, x') \in A_{i,j} \text{ and } a \in \mathcal{A}^{q,Q}(t_i). \end{aligned} \quad (4.1)$$

Now, let $(a, \beta) \in \mathcal{A}^{q,Q}(t) \times \mathcal{B}(t)$ be arbitrary. We construct the strategy β^n by

$$\beta^n(a)_s := \mathbf{1}_{[t,\theta]}(s)\beta(a)_s + \mathbf{1}_{(\theta,T]}(s) \left(\mathbf{1}_{\mathcal{U} \setminus A^n}(\theta, X_\theta)\beta(a)_s + \sum_{1 \leq i,j \leq n} \mathbf{1}_{A_{i,j}}(\theta, X_\theta)\beta_{i,j}(a|_{(t_i,T]})_s \right)$$

where it is understood that $X := X^{t,x;a,\beta(a)}$ and $\theta := \theta^{a,\beta(a)}$. An application of the tower property yields

$$J(t, x; a, \beta^n(a)) = \mathbb{E} \left[J_0(t, x; a, \beta(a); \theta) \mathbf{1}_{A^n} + J(t, x; a, \beta(a)) \mathbf{1}_{\mathcal{U} \setminus A^n} + \sum_{1 \leq i,j \leq n} J(\theta, X_\theta; a|_{(t_i,T]}, \beta_{i,j}(a|_{(t_i,T]})) \mathbf{1}_{A_{i,j}} \right]$$

where we have omitted the argument (θ, X_θ) to the indicator functions for brevity. By (4.1), we immediately get

$$\mathbb{E} \left[\sum_{1 \leq i,j \leq n} J(\theta, X_\theta; a|_{(t_i,T]}, \beta_{i,j}(a|_{(t_i,T]})) \mathbf{1}_{A_{i,j}}(\theta, X_\theta) \right] \leq \mathbb{E} [\varphi(\theta, X_\theta) \mathbf{1}_{A^n}(\theta, X_\theta)] + \epsilon/2.$$

Since φ is bounded and $X_\theta \in \cup_{i \geq 1} \cup_{j \geq 1} A_{i,j}$ \mathbb{P} -almost surely, we can apply the dominated convergence theorem (DCT) to yield

$$\mathbb{E} [\varphi(\theta, X_\theta) \mathbf{1}_{A^n}(\theta, X_\theta)] \rightarrow \mathbb{E} [\varphi(\theta, X_\theta)] \text{ as } n \rightarrow \infty.$$

The DCT also yields, due to the boundedness of J by Lemma 3.4,

$$\mathbb{E} [J(t, x; a, \beta(a)) \mathbf{1}_{\mathcal{U} \setminus A^n}(\theta, X_\theta)] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The same argument can be made to get

$$\mathbb{E} [J_0(t, x; a, \beta(a); \theta) \mathbf{1}_{A^n}(\theta, X_\theta)] \rightarrow \mathbb{E} [J_0(t, x; a, \beta(a); \theta)] \text{ as } n \rightarrow \infty.$$

It follows that there exists an n_0 (independent of $a \in \mathcal{A}^{q,Q}(t)$) such that

$$J(t, x; a, \beta^{n_0}(a)) \leq \mathbb{E} [J_0(t, x; a, \beta(a); \theta) + \varphi(\theta, X_\theta)] + \epsilon. \quad (4.2)$$

Let $(\varphi_m)_m$ be a sequence of continuous functions converging monotonically to $(v^+)^*$ from above. Note that we can pick the elements of our sequence to be bounded. By the DCT,

$$\mathbb{E} [\varphi_m(\theta, X_\theta)] \rightarrow \mathbb{E} [(v^+)^*(\theta, X_\theta)] \text{ as } m \rightarrow \infty.$$

It follows that we can replace φ appearing in the bound (4.2) by $(v^+)^*$. Since

$$v^+(t, x) \leq \sup_{a \in \mathcal{A}^{q,Q}(t)} J(t, x; a, \beta^{n_0}(a)) \leq \sup_{a \in \mathcal{A}^{q,Q}(t)} \mathbb{E} [J_0(t, x; a, \beta(a); \theta) + (v^+)^*(\theta, X_\theta)] + \epsilon,$$

the result follows by recalling that β and ϵ were arbitrary so that ϵ can be safely removed from the inequality and infimums of both sides can be taken. \blacksquare

5 Dynamic programming equation

We prove, in this section, Theorem 2.11. We first give a lemma, which appears (in slightly different flavours) in [24, Proposition 2.3], [27, Lemma 5.1], [31, Lemma 4.3], and possibly elsewhere. We provide a proof for completeness.

Lemma 5.1. *Let $u, w : \text{cl } \mathcal{O} \rightarrow \mathbb{R}$ be bounded. \mathcal{M} is monotone: if $u \geq w$ pointwise, $\mathcal{M}u \geq \mathcal{M}w$ pointwise. Moreover, $\mathcal{M}u_*$ (resp. $\mathcal{M}u^*$) is lower (resp. upper) semicontinuous and $\mathcal{M}u_* \leq (\mathcal{M}u)_*$ (resp. $(\mathcal{M}u)^* \leq \mathcal{M}u^*$).*

Proof. The monotonicity property follows directly from the definition.

Let $\epsilon > 0$ and let $(t_n, x_n)_n$ be a $\text{cl } \mathcal{O}$ -valued sequence converging to some (t, x) . Pick $z^\epsilon \in Z$ such that $u_*(t, x + \Gamma(t, z^\epsilon)) + K(t, z^\epsilon) + \epsilon \geq \mathcal{M}u_*(t, x)$. Then,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mathcal{M}u_*(t_n, x_n) &\geq \liminf_{n \rightarrow \infty} \{u_*(t_n, x_n + \Gamma(t_n, z^\epsilon)) + K(t_n, z^\epsilon)\} \\ &\geq u_*(t, x + \Gamma(t, z^\epsilon)) + K(t, z^\epsilon) \geq \mathcal{M}u_*(t, x) - \epsilon. \end{aligned}$$

Since ϵ was arbitrary, it follows that $\mathcal{M}u_*$ is lower semicontinuous. By monotonicity, we have $\mathcal{M}u \geq \mathcal{M}u_*$ pointwise, so that we can take lower semicontinuous envelopes of both sides to get $(\mathcal{M}u)_* \geq (\mathcal{M}u_*)_* = \mathcal{M}u_*$ as desired.

Lastly, let $(t_n, x_n)_n$ be a $\text{cl } \mathcal{O}$ -valued sequence converging to some (t, x) . Due to the upper semicontinuity and boundedness of u^* and the growth conditions on K (Assumption 2.7 (iii)), there exists a compact set $Q \subset Z$ such that for each n , there is a $z_n \in Q$ with $\mathcal{M}u^*(t_n, x_n) = u^*(t_n, x_n + \Gamma(t_n, z_n)) + K(t_n, z_n)$. Therefore, $(z_n)_n$ admits a convergent subsequence with limit $\hat{z} \in Z$ and hence

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathcal{M}u^*(t_n, x_n) &= \limsup_{n \rightarrow \infty} \{u^*(t_n, x_n + \Gamma(t_n, z_n)) + K(t_n, z_n)\} \\ &\leq u^*(t, x + \Gamma(t, \hat{z})) + K(t, \hat{z}) \leq \mathcal{M}u^*(t, x). \end{aligned}$$

The rest of the proof is similar to previous arguments. ■

We are now ready to prove Theorem 2.11. The boundary conditions follow from Assumption 2.8 (ii) (see also [31, Proof of Theorem 4.2]), and are thus ignored below. In the proofs, we write \mathcal{A} and \mathcal{A} in lieu of $\mathcal{A}^{q, Q}$ and $\mathcal{A}^{q, Q}$ for brevity.

We begin by showing that v^+ is a subsolution.

Proof of Theorem 2.11 (subsolution). Without loss of generality, let $(t, x, \varphi) \in \mathcal{O} \times C^{1,2}(\mathcal{O})$ be such that $((v^+)^* - \varphi)(t, x) = 0$ is a *strict* local maximum of $(v^+)^* - \varphi$. By Lemma A.1 of Appendix A, we can also assume φ is compactly supported. If $((v^+)^* - \mathcal{M}(v^+)^*)(t, x) \leq 0$, the claim is trivial, so we assume the negation of this inequality. To complete the proof, we assume

$$\inf_{b \in B} \left\{ (\partial_t + \mathbb{L}^b) \varphi(t, x) + f(t, x, b) \right\} < 0$$

and show that this contradicts the first inequality appearing in the DPP of Theorem 2.9. Note that the above implies

$$(\partial_t + \mathbb{L}^b) \varphi(t, x) + f(t, x, b) < 0$$

for some $b \in B$, which we leave fixed for the remainder. By Lemma 5.1, $\varphi - \mathcal{M}(v^+)^*$ is lower semicontinuous. Since for some $\delta > 0$,

$$(\varphi - \mathcal{M}(v^+)^*)(t, x) = ((v^+)^* - \mathcal{M}(v^+)^*)(t, x) \geq 4\delta,$$

we can find $h > 0$ (by lower semicontinuity) such that $t + h < T$ and

$$\begin{aligned} ((v^+)^* - \varphi)(t, x) \text{ is a strict maximum of } (v^+)^* - \varphi, \\ \varphi - \mathcal{M}(v^+)^* \geq 3\delta, \text{ and } (\partial_t + \mathbb{L}^b)\varphi + f^b \leq 0 \text{ on } \text{cl } \mathcal{N}_{2h}, \\ \text{where } \mathcal{N}_h := ((t - h, t + h) \times B(x; h)) \cap \mathcal{O}. \end{aligned}$$

Let (t_n, x_n) be a sequence converging to (t, x) with $v^+(t_n, x_n) \rightarrow (v^+)^*(t, x)$. We henceforth suppose n is sufficiently large for $(t_n, x_n) \in \mathcal{N}_h \subset \mathcal{N}_{2h}$ to hold. With a slight abuse of notation, we also use b to refer to a constant control in $\mathcal{B}(t_n)$ taking on the value $b \in B$. Since the inf-player is free to play a constant strategy returning this control, the DPP implies that for a nonanticipative family $\{\theta^a\}_{a \in \mathcal{A}(t_n)}$ of \mathbb{Q}_{t_n} -valued $(\mathcal{F}_{t,s})_{s \in [t_n, T]}$ -stopping times,

$$v^+(t_n, x_n) \leq \sup_{a \in \mathcal{A}(t_n)} \mathbb{E} \left[\int_{t_n}^{\theta} f(s, X_s^n, b) ds + \sum_{\tau_j \leq \theta} K(\tau_j, z_j) + (v^+)^*(\theta, X_\theta^n) \right]$$

where it is understood that $X^n := X^{t_n, x_n; a, b}$ and $\theta := \theta^a$. In particular, if $\theta \leq \tau_1$ (the time of first impulse), it follows immediately from the suboptimality of multiple impulses at the same time (Assumption 2.8 (i)) that

$$\begin{aligned} v^+(t_n, x_n) &\leq \sup_{\tau_1 \in \mathcal{T}_n} \mathbb{E} \left[\int_{t_n}^{\theta} f(s, X_s^n, b) ds + K(\tau_1, z_1) \mathbf{1}_{\{\theta = \tau_1\}} + (v^+)^*(\theta, X_\theta^n) \right] \\ &\leq \sup_{\tau_1 \in \mathcal{T}_n} \mathbb{E} \left[\int_{t_n}^{\theta} f(s, X_s^n, b) ds + (v^+)^*(\theta, X_\theta^n) \mathbf{1}_{\{\theta < \tau_1\}} \right. \\ &\quad \left. + \mathcal{M}(v^+)^*(\theta, X_{\theta-}^n) \mathbf{1}_{\{\theta = \tau_1\}} \right] \end{aligned} \quad (5.1)$$

where \mathcal{T}_n is the set of all $\mathbb{Q}_{t_n} \cup \{+\infty\}$ -valued $(\mathcal{F}_{t_n, s})_{s \in [t_n, T]}$ -stopping times. Now, fix $\tau_1 \in \mathcal{T}_n$. Let

$$\theta_n := \inf \{s > t_n : (s, X_s^n) \notin \mathcal{N}_h\} \text{ and } \theta_n^m := \tau_1 \wedge \min(\{kT/m : k \geq 1\} \cap [\theta_n, T]) \quad (5.2)$$

where m is a positive integer. Note, in particular, that for m large enough, θ_n^m is \mathbb{Q}_{t_n} -valued. Since (t, x) is a strict maximum point,

$$-3\gamma := \max_{\text{cl } \mathcal{N}_{2h} \setminus \text{int } \mathcal{N}_h} ((v^+)^* - \varphi) < 0.$$

Let

$$\eta_n := \varphi(t_n, x_n) - v^+(t_n, x_n) \geq 0$$

and note that $\eta_n \rightarrow 0$ as $n \rightarrow \infty$. Now, let $A_m := \{(\theta_n^m, X_{\theta_n^m}^n) \in \mathcal{N}_{2h}\}$ and pick m large enough so that

$$\mathbb{E} \left[\left(\int_{t_n}^{\theta_n^m} -(\partial_t + \mathbb{L}^b)\varphi(s, X_s^n) ds + \varphi(\theta_n^m, X_{\theta_n^m-}^n) \right) \mathbf{1}_{\Omega \setminus A_m} \right] \geq -(\gamma \wedge \delta)$$

and

$$0 \geq \mathbb{E} \left[\left(\int_{t_n}^{\theta_n^m} f(s, X_s^n, b) ds + (v^+)^*(\theta_n^m, X_{\theta_n^m}^n) \mathbf{1}_{\{\theta_n^m < \tau_1\}} \right. \right. \\ \left. \left. + \mathcal{M}(v^+)^*(\theta_n^m, X_{\theta_n^m-}^n) \mathbf{1}_{\{\theta_n^m = \tau_1\}} + 3(\gamma \wedge \delta) \right) \mathbf{1}_{\Omega \setminus A_m} \right] - (\gamma \wedge \delta).$$

This is possible since f , $(v^+)^*$, and $\mathcal{M}(v^+)^*$ are bounded (recall the growth conditions of Assumption 2.7 (iii)), φ has compact support, and $\mathbf{1}_{\Omega \setminus A_m} \rightarrow 0$ \mathbb{P} -almost surely as $m \rightarrow \infty$. We can now apply Dynkin's formula to get

$$\begin{aligned} v^+(t_n, x_n) + \eta_n &= \varphi(t_n, x_n) \\ &= \mathbb{E} \left[\left(\int_{t_n}^{\theta_n^m} -(\partial_t + \mathbf{L}^b) \varphi(s, X_s^n) ds + \varphi(\theta_n^m, X_{\theta_n^m-}^n) \right) (\mathbf{1}_{A_m} + \mathbf{1}_{\Omega \setminus A_m}) \right] \end{aligned}$$

and hence by applying the previous inequalities,

$$\begin{aligned} v^+(t_n, x_n) &\geq \mathbb{E} \left[\left(\int_{t_n}^{\theta_n^m} f(s, X_s^n, b) ds + (v^+)^*(\theta_n^m, X_{\theta_n^m-}^n) \mathbf{1}_{\{\theta_n^m < \tau_1\}} \right) \mathbf{1}_{A_m} \right] - (\gamma \wedge \delta) \\ &\geq \mathbb{E} \left[\int_{t_n}^{\theta_n^m} f(s, X_s^n, b) ds + (v^+)^*(\theta_n^m, X_{\theta_n^m-}^n) \mathbf{1}_{\{\theta_n^m < \tau_1\}} \right] + (\gamma \wedge \delta) \\ &\quad + \mathcal{M}(v^+)^*(\theta_n^m, X_{\theta_n^m-}^n) \mathbf{1}_{\{\theta_n^m = \tau_1\}} \end{aligned}$$

Now, pick n sufficiently large so that $(\gamma \wedge \delta) - \eta_n > 0$. Taking supremums, we get

$$v^+(t_n, x_n) \geq (\gamma \wedge \delta) - \eta_n + \sup_{\tau_1 \in \mathcal{T}_n} \mathbb{E} \left[\int_{t_n}^{\theta_n^m} f(s, X_s^n, b) ds + (v^+)^*(\theta_n^m, X_{\theta_n^m-}^n) \mathbf{1}_{\{\theta_n^m < \tau_1\}} \right] + \mathcal{M}(v^+)^*(\theta_n^m, X_{\theta_n^m-}^n) \mathbf{1}_{\{\theta_n^m = \tau_1\}}$$

thereby contradicting (5.1). ■

We now show that v^- is a supersolution. In the proof below, we begin by showing $(v^-)_* \geq \mathcal{M}(v^-)_*$ on $\text{cl } \mathcal{O}$. If the impulse times were not restricted to rational numbers, this claim would be trivial. However, the argument below is slightly more delicate than usual.

Proof of Theorem 2.11 (supersolution). Let $(t, x) \in \text{cl } \mathcal{O}$. Let $(t_n)_n$ be a nonincreasing sequence taking values in \mathbb{Q}_t and converging to t . Note that for each $z \in Z$, the sup-player is free to play a control $a \in \mathcal{A}(t)$ with a single impulse at $\tau_1 = t_n$ with $z_1 = z$. Therefore, the DPP (with $\theta = t_n$) implies that for each n , there exists a $b_n \in \mathcal{B}(t)$ such that

$$v^-(t, x) \geq \mathbb{E} \left[(v^-)_*(t_n, X_{t_n-}^n + \Gamma(t_n, z)) + K(t_n, z) \right] - (t_n - t) \|f\|_\infty - 1/n$$

where $X^n := X^{t, x; a, b_n}$. Note that (see, e.g., the proof of [34, Theorem 2.4] for details)

$$\mathbb{E} \left[|X_{t_n-}^n - x|^2 \right] \leq \text{const.} (1 + |x|^2) (t_n - t),$$

implying $X_{t_n-}^n \rightarrow x$ in $\mathbb{L}^2(\Omega_{t, T})$. Therefore, we can extract a subsequence, which with a slight abuse of notation we also refer to as $(X_{t_n-}^n)_n$, such that $X_{t_n-}^n \rightarrow x$ \mathbb{P} -almost surely. Since $(v^-)_*$ and $t \mapsto K(t, z)$ are bounded ($[t, T]$ is compact and K is continuous), we can apply Fatou's lemma to get

$$\begin{aligned} v^-(t, x) &\geq \mathbb{E} \left[\liminf_{n \rightarrow \infty} \left\{ (v^-)_*(t_n, X_{t_n-}^n + \Gamma(t_n, z)) + K(t_n, z) \right\} \right] \\ &\geq (v^-)_*(t, x + \Gamma(t, z)) + K(t, z). \end{aligned}$$

Taking supremums, we get $v^-(t, x) \geq \mathcal{M}(v^-)_*(t, x)$. Because this inequality holds on $\text{cl } \mathcal{O}$, we have $(v^-)_* \geq (\mathcal{M}(v^-)_*)_* = \mathcal{M}(v^-)_*$ by Lemma 5.1.

Let $(t, x, \varphi) \in \mathcal{O} \times C^{1,2}(\mathcal{O})$ be such that $((v^-)_* - \varphi)(t, x) = 0$ is a *strict* local minimum of $(v^-)_* - \varphi$. By Lemma A.1 of Appendix A, we can also assume φ is compactly supported. To complete the proof, we assume

$$\inf_{b \in B} \left\{ (\partial_t + \mathbb{L}^b) \varphi(t, x) + f(t, x, b) \right\} > 0$$

and show that this contradicts the second inequality appearing in the DPP of Theorem 2.9. By continuity, we can find $h > 0$ such that $t + h < T$ and

$((v^-)_* - \varphi)(t, x)$ is a strict minimum of $(v^-)_* - \varphi$

$$\text{and } \inf_{b \in B} \left\{ (\partial_t + \mathbb{L}^b) \varphi + f^b \right\} \geq 0 \text{ on } \text{cl } \mathcal{N}_{2h},$$

where $\mathcal{N}_h := ((t - h, t + h) \times B(x; h)) \cap \mathcal{O}$.

Let (t_n, x_n) be a sequence converging to (t, x) with $v^-(t_n, x_n) \rightarrow (v^-)_*(t, x)$. We henceforth suppose n is sufficiently large for $(t_n, x_n) \in \mathcal{N}_h \subset \mathcal{N}_{2h}$ to hold. Since the sup-player can play a constant strategy returning the control $\hat{a} \in \mathcal{A}(t_n)$ with no impulses, the DPP implies that for a nonanticipative family $\{\theta^b\}_{b \in \mathcal{B}(t)}$ of \mathbb{Q}_{t_n} -valued $(\mathcal{F}_{t,s})_{s \in [t_n, T]}$ -stopping times,

$$v^-(t_n, x_n) \geq \inf_{b \in \mathcal{B}(t_n)} \mathbb{E} \left[\int_{t_n}^{\theta^b} f(s, X_s^n, b_s) ds + (v^-)_*(\theta, X_\theta^n) \right] \quad (5.3)$$

where it is understood that $X^n := X^{t_n, x_n; \hat{a}, b}$ and $\theta := \theta^b$. Now, fix $b \in \mathcal{B}(t)$. Let

$$\theta_n := \inf \{s > t_n : (s, X_s^n) \notin \mathcal{N}_h\} \text{ and } \theta_n^m := \min(\{kT/m : k \geq 1\} \cap [\theta_n, T])$$

where m is a positive integer. Note, in particular, that for m large enough, θ_n^m is \mathbb{Q}_{t_n} -valued. Since (t, x) is a strict minimum point,

$$3\gamma := \min_{\text{cl } \mathcal{N}_{2h} \setminus \text{int } \mathcal{N}_h} ((v^-)_* - \varphi) > 0.$$

Let

$$\eta_n := v^-(t_n, x_n) - \varphi(t_n, x_n) \geq 0$$

and note that $\eta_n \rightarrow 0$ as $n \rightarrow \infty$. Now, let $A_m := \{(\theta_n^m, X_{\theta_n^m}^n) \in \mathcal{N}_{2h}\}$ and pick m large enough so that

$$\mathbb{E} \left[\left(\int_{t_n}^{\theta_n^m} -(\partial_t + \mathbb{L}^{b_s}) \varphi(s, X_s^n) ds + \varphi(\theta_n^m, X_{\theta_n^m}^n) \right) \mathbf{1}_{\Omega \setminus A_m} \right] \leq \gamma$$

and

$$0 \leq \mathbb{E} \left[\left(\int_{t_n}^{\theta_n^m} f(s, X_s^n, b_s) ds + (v^-)_*(\theta_n^m, X_{\theta_n^m}^n) - 3\gamma \right) \mathbf{1}_{\Omega \setminus A_m} \right] + \gamma.$$

This is possible since f and $(v^-)_*$ are bounded, φ has compact support, and $\mathbf{1}_{\Omega \setminus A_m} \rightarrow 0$ \mathbb{P} -almost surely as $m \rightarrow \infty$. We can now apply Dynkin's formula to get

$$\begin{aligned}
v^-(t_n, x_n) - \eta_n &= \varphi(t_n, x_n) \\
&= \mathbb{E} \left[\left(\int_{t_n}^{\theta_n^m} -(\partial_t + \mathbb{L}^{b_s})\varphi(s, X_s^n) ds + \varphi(\theta_n^m, X_{\theta_n^m}^n) \right) (\mathbf{1}_{A_m} + \mathbf{1}_{\Omega \setminus A_m}) \right] \\
&\leq \mathbb{E} \left[\left(\int_{t_n}^{\theta_n^m} f(s, X_s^n, b_s) ds + (v^-)_*(\theta_n^m, X_{\theta_n^m}^n) - 3\gamma \right) \mathbf{1}_{A_m} \right] + \gamma \\
&\leq \mathbb{E} \left[\int_{t_n}^{\theta_n^m} f(s, X_s^n, b_s) ds + (v^-)_*(\theta_n^m, X_{\theta_n^m}^n) \right] - \gamma.
\end{aligned}$$

Now, pick n sufficiently large so that $\eta_n - \gamma < 0$. Taking infimums, we get

$$v^-(t_n, x_n) \leq \eta_n - \gamma + \inf_{b \in \mathcal{B}(t_n)} \mathbb{E} \left[\int_{t_n}^{\theta_n^m} f(s, X_s^n, b_s) ds + (v^-)_*(\theta_n^m, X_{\theta_n^m}^n) \right],$$

thereby contradicting (5.3). ■

Remark. Though the collection $(\mathcal{B}(t))_{t \in [0, T]}$ is quite rich, the proofs of the DPP and DPE need only the collection $(\mathcal{B}_0(t))_{t \in [0, T]}$ of piecewise constant controls. Therefore, we are free to restrict each $\mathcal{B}(t)$ (so long as $\mathcal{B}_0(t) \subset \mathcal{B}(t)$) without invalidating our results.

As promised, we give below an example in which Assumption 2.8 (ii) is satisfied.

Example 5.2 (Terminal continuity). Suppose there exists a time $t_0 < T$ after which it is provably suboptimal to perform an impulse (e.g., consider the case of $K(t, z)$ sufficiently large in magnitude for all $(t, z) \in [t_0, T] \times Z$). Let $x \in \mathbb{R}^d$ and $(t_n, x_n)_n$ be a sequence converging to (T, x) . Then, $v^+(t_n, x_n) \rightarrow g(x)$ and $v^-(t_n, x_n) \rightarrow g(x)$ so that Assumption 2.8 (ii) is trivially satisfied.

To see why this is the case, for fixed $b \in \mathcal{B}(t_n)$, let $X^n := X^{t_n, x_n; \hat{a}, b}$ where $\hat{a} \in \mathcal{A}(t_n)$ is a control with no impulses. Without loss of generality, we can assume n is sufficiently large so that $t_n \geq t_0$. Then,

$$\begin{aligned}
|v^+(t_n, x_n) - g(x)| &\leq \sup_{b \in \mathcal{B}(t_n)} \mathbb{E} \left[\int_{t_n}^T |f(s, X_s^n, b_s)| ds + |g(X_T^n) - g(x)| \right] \\
&\leq \|f\|_\infty (T - t_n) + \text{const.} \sup_{b \in \mathcal{B}(t_n)} \mathbb{E} [|X_T^n - x|].
\end{aligned}$$

Moreover, by an argument using Grönwall's lemma and the linear growth of μ and σ (see, e.g., the proof of [34, Theorem 2.4] for details),

$$\mathbb{E} [|X_T^n - x|] \leq \mathbb{E} [|X_{t_n}^n - x_n|] + |x_n - x| \leq \text{const.} (1 + |x_n|^2) (T - t_n) + |x_n - x|.$$

Since g is Lipschitz, combining the above inequalities and taking $n \rightarrow \infty$ yields the desired result. The proof for the case of v^- is identical.

6 Comparison principle

We prove, in this section, Theorem 2.12. We first prepare a few lemmas. The result below, which also appears in [31, Lemma 5.5], follows directly from sup-manipulations.

Lemma 6.1. *Let $u, w : \text{cl } \mathcal{O} \rightarrow \mathbb{R}$ be bounded. \mathcal{M} is convex:*

$$\mathcal{M}(\lambda u + (1 + \lambda)w) \leq \lambda \mathcal{M}u + (1 + \lambda) \mathcal{M}w \text{ for } 0 \leq \lambda \leq 1.$$

Moreover, \mathcal{M} is “anti-convex”:

$$\mathcal{M}(-\lambda u + (1 + \lambda)w) \geq -\lambda \mathcal{M}u + (1 + \lambda) \mathcal{M}w \text{ for } \lambda > 0.$$

We now perform a change of variables to introduce a positive “discount” term $\rho > 0$. Concretely, let

$$F_\rho(\cdot, u, Du(\cdot), D^2u(\cdot)) := \begin{cases} \min\{-\inf_{b \in B}\{(\partial_t + \mathbb{L}^b - \rho)u + f_\rho^b\}, u - \mathcal{M}_\rho u\} & \text{on } \mathcal{O} \\ \min\{u - g_\rho, u - \mathcal{M}_\rho u\} & \text{on } \partial^+ \mathcal{O} \end{cases} \quad (6.1)$$

where $f_\rho^b := f_\rho(t, x, b)$, $f_\rho := e^{\rho t} f$, $g_\rho := e^{\rho t} g$, $K_\rho := e^{\rho t} K$, and

$$\mathcal{M}_\rho u(t, x) := \sup_{z \in Z} \{u(t, x + \Gamma(t, z)) + K_\rho(t, z)\}.$$

If u is a subsolution of $F = 0$, $e^{\rho t} u$ is a subsolution of $F_\rho = 0$. The same claim holds for supersolutions. Therefore, we need only consider uniqueness under $\rho > 0$, which we pick arbitrarily and leave fixed for the remainder of this section. Note that Lemmas 5.1 and 6.1 remain valid for \mathcal{M}_ρ .

The following lemma allows us to construct a family of “strict” supersolutions of $F_\rho = 0$ by taking combinations of an ordinary supersolution and a specific constant. This technique appears in [24, Lemma 3.2], and is needed due to the implicit form of the obstacle.

Lemma 6.2. *Let w be a supersolution of (6.1), $c := \max\{(\|f_\rho\|_\infty + 1)/\rho, \|g_\rho\|_\infty + 1\}$, and $\xi := \min\{1, K_0\}$. Then, for each $0 < \lambda < 1$, $w_\lambda := (1 - \lambda)w + \lambda c$ is a supersolution of*

$$F_\rho(\cdot, u, Du(\cdot), D^2u(\cdot)) - \lambda \xi = 0 \text{ on } \text{cl } \mathcal{O}. \quad (6.2)$$

Proof. Below, we treat c both as a constant and a constant function on $\text{cl } \mathcal{O}$ taking the value c . First, note that for $(t, x) \in \mathcal{O}$,

$$\begin{aligned} \min \left\{ -\inf_{b \in B} \left\{ (\partial_t + \mathbb{L}^b - \rho)c(t, x) + f_\rho(t, x, b) \right\}, c(t, x) - \mathcal{M}_\rho c(t, x) \right\} \\ \geq \min \left\{ -(-\rho c + \|f_\rho\|_\infty), K_0 \right\} \geq \xi. \end{aligned}$$

Similarly, for $(t, x) \in \partial^+ \mathcal{O}$,

$$\min \{c(t, x) - g_\rho(x), (c - \mathcal{M}_\rho c)(t, x)\} \geq \min \{c - \|g_\rho\|_\infty, K_0\} \geq \xi.$$

Note that we have proved that c is a *classical* supersolution of $F_\rho - \xi = 0$.

Without loss of generality, we assume that w is lower semicontinuous (otherwise, replace w by its lower semicontinuous envelope). Now, let $(t, x, \varphi_\lambda) \in \mathcal{O} \times C^{1,2}(\mathcal{O})$ be such that

$(w_\lambda - \varphi_\lambda)(t, x) = 0$ is a local minimum of $w_\lambda - \varphi_\lambda$. Further letting $\lambda' := 1 - \lambda$ for brevity and $\varphi := (\varphi_\lambda - \lambda c)/\lambda'$, it follows that (t, x) is also a local minimum point of $w - \varphi$ since

$$\lambda' (w - \varphi) = \lambda' (w - (\varphi_\lambda - \lambda c)/\lambda') = \lambda' w + \lambda c - \varphi_\lambda = w_\lambda - \varphi_\lambda.$$

We now seek to show that

$$-\inf_{b \in B} \left\{ (\partial_t + \mathbb{L}^b - \rho) \varphi_\lambda(t, x) + f_\rho(t, x, b) \right\} \geq \lambda \xi,$$

for which it is sufficient to show that for some choice of $b \in B$,

$$(\partial_t + \mathbb{L}^b - \rho) \varphi_\lambda(t, x) + f_\rho(t, x, b) \leq -\lambda \xi.$$

In particular, using the supersolution property of w along with the continuity of φ and compactness of B , there exists $b \in B$ such that

$$0 \geq \lambda' \left((\partial_t + \mathbb{L}^b - \rho) \varphi(t, x) + f_\rho(t, x, b) \right) \geq (\partial_t + \mathbb{L}^b - \rho) \varphi_\lambda(t, x) + f_\rho(t, x, b) + \lambda \xi.$$

On $\text{cl } \mathcal{O}$, since w is a supersolution, we have that $w \geq \mathcal{M}_\rho w$. Along with the convexity of \mathcal{M}_ρ (Lemma 6.1), this yields

$$w_\lambda - \mathcal{M}_\rho w_\lambda \geq w_\lambda - \lambda' \mathcal{M}_\rho w - \lambda \mathcal{M}_\rho c \geq w_\lambda - \lambda' w - \lambda \mathcal{M}_\rho c = \lambda(c - \mathcal{M}_\rho c) \geq \lambda \xi.$$

Lastly, on $\partial^+ \mathcal{O}$, we have

$$w_\lambda - g_\rho = \lambda' (w - g_\rho) + \lambda (c - g_\rho) \geq \lambda \xi,$$

so that w_λ satisfies the boundary condition. ■

We give a result that describes the regularity of the “non-impulse” part of the HJBI (6.1). In the lemma statement, I_d denotes the identity matrix in $\mathbb{R}^{d \times d}$.

Lemma 6.3. *Let $D \subset \mathbb{R}^d$ be compact and H be given by*

$$H(t, x, r, q, X) := -\inf_{b \in B} \left\{ \frac{1}{2} \text{trace}(\sigma(x, b) \sigma^\top(x, b) X) + \langle \mu(x, b), q \rangle - \rho r + f_\rho(t, x, b) \right\}. \quad (6.3)$$

Then, there exists a modulus of continuity ω such that for all $(t, x, r, X), (s, y, r', Y) \in [0, T] \times D \times \mathbb{R} \times \mathcal{S}(d)$ satisfying

$$\begin{pmatrix} X & \\ & -Y \end{pmatrix} \preceq \text{const.} \alpha \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}$$

and all positive constants α and ϵ ,

$$\begin{aligned} & H(s, y, r', \alpha(x - y) - \epsilon y, Y - 2\epsilon I_d) - H(t, x, r, \alpha(x - y) + \epsilon x, X + 2\epsilon I_d) \\ & \leq \rho(r' - r) + \text{const.} (\alpha |x - y|^2 + \epsilon(1 + |x|^2 + |y|^2)) + \omega(|(t, x) - (s, y)|). \end{aligned}$$

Proof. Let $M(b) := \sigma(x, b)$ and $N(b) := \sigma(y, b)$. First, note that

$$\begin{aligned} & H(s, y, r', \alpha(x - y) - \epsilon y, Y - 2\epsilon I_d) - H(t, x, r, \alpha(x - y) + \epsilon x, X + 2\epsilon I_d) \\ & \leq \sup_{b \in B} \left\{ \begin{aligned} & \text{trace}(M(b) M(b)^\top (X + 2\epsilon I_d) - N(b) N(b)^\top (Y - 2\epsilon I_d)) \\ & + \alpha \langle \mu(x, b) - \mu(y, b), x - y \rangle + \epsilon \langle \mu(x, b), x \rangle + \epsilon \langle \mu(y, b), y \rangle \\ & + \rho(r - r') + f_\rho(t, x, b) - f_\rho(s, y, b) \end{aligned} \right\}. \quad (6.4) \end{aligned}$$

Omitting the dependence on b for brevity and employing the linear growth of μ and the inequality $|x| \leq 1 + |x|^2$,

$$\epsilon \langle \mu(x), x \rangle + \epsilon \langle \mu(y), y \rangle \leq \text{const.} \epsilon ((1 + |x|)|x| + (1 + |y|)|y|) \leq \text{const.} \epsilon (1 + |x|^2 + |y|^2).$$

Denoting by $\|\cdot\|_F$ the Frobenius norm, the linear growth of σ similarly yields

$$\epsilon \text{trace}(MM^\top) + \epsilon \text{trace}(NN^\top) = \epsilon \|M\|_F^2 + \epsilon \|N\|_F^2 \leq \text{const.} \epsilon (1 + |x|^2 + |y|^2).$$

We also have the inequalities

$$\alpha \langle \mu(x) - \mu(y), x - y \rangle \leq \alpha |\mu(x) - \mu(y)| |x - y|$$

and

$$\begin{aligned} \text{trace}(MM^\top X - NN^\top Y) &= \text{trace} \left(\begin{pmatrix} MM^\top & MN^\top \\ NM^\top & NN^\top \end{pmatrix} \begin{pmatrix} X \\ -Y \end{pmatrix} \right) \\ &\leq \text{const.} \alpha \text{trace} \left(\begin{pmatrix} MM^\top & MN^\top \\ NM^\top & NN^\top \end{pmatrix} \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} \right) \\ &= \text{const.} \alpha \text{trace}((M - N)(M - N)^\top) \\ &= \text{const.} \alpha \|M - N\|_F^2. \end{aligned}$$

The desired result follows by applying the above inequalities to (6.4) and invoking the uniform continuity of f on the compact set $[0, T] \times D \times B$ and the Lipschitzness of μ and σ . \blacksquare

The following appears in [25, Problem 2.4.17].

Lemma 6.4. *Let $(a_n)_n$ and $(b_n)_n$ be sequences of nonnegative numbers. If a_n converges to a positive number a , $\limsup_{n \rightarrow \infty} a_n b_n = a \limsup_{n \rightarrow \infty} b_n$.*

We are finally ready to prove the comparison principle. We mention that we cannot directly use the “parabolic” Crandall-Ishii lemma [15, Theorem 8.3] in the proof since condition (8.5) of [15] cannot be satisfied due to the impulse term (see also the proof of [4, Lemma 11.1] for a more in-depth discussion of this issue). We rely instead on the “elliptic” Crandall-Ishii lemma [15, Theorem 3.2] and employ a variable-doubling argument inspired by [14, Lemma 8]. Throughout the proof, we use the symbol $\text{cl}(\mathcal{P}_O^{2,\pm} u)$ to denote a limiting parabolic semijet of u , whose definition is given in [15, §8].

Proof of Theorem 2.12. Let u be a bounded subsolution and w be a bounded supersolution of (6.1). As in the proof of Lemma 6.2, we can assume that u (resp. w) is upper (resp. lower) semicontinuous (otherwise, replace u and w by their semicontinuous envelopes). Let c be given as in Lemma 6.2 and $w_m := (1 - 1/m)w + c/m$ for all integers $m > 1$. Note that

$$\sup_O \{u - w_m\} = \sup_O \{u - w + (w - c)/m\} \geq \sup_O \{u - w\} - (\|w\|_\infty + c)/m.$$

Therefore, to prove the comparison principle, it is sufficient to show $u - w_m \leq 0$ (pointwise) along a subsequence of $(w_m)_m$. We establish it for all m .

To that end, fix m and suppose $\delta := \sup_O \{u - w_m\} > 0$. Letting $\nu > 0$, we can find $(t^\nu, x^\nu) \in O$ such that $(u - w_m)(t^\nu, x^\nu) \geq \delta - \nu$. Let

$$\varphi(t, x, s, y) := \frac{\alpha}{2} (|t - s|^2 + |x - y|^2) + \frac{\epsilon}{2} (|x|^2 + |y|^2)$$

be a smooth function parameterized by constants $\alpha > 0$ and $0 < \epsilon \leq 1$. Further let $\Phi(t, x, s, y) := u(t, x) - w_m(s, y) - \varphi(t, x, s, y)$ and note that

$$\begin{aligned} \sup_{(t, x, s, y) \in ([0, T] \times \mathbb{R}^d)^2} \Phi(t, x, s, y) &\geq \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \left\{ (u - w_m)(t, x) - \epsilon |x|^\nu \right\} \\ &\geq (u - w_m)(t^\nu, x^\nu) - \epsilon |x^\nu|^\nu \\ &\geq \delta - \nu - \epsilon |x^\nu|^\nu. \end{aligned}$$

We henceforth assume ν and ϵ are small enough (e.g., pick $\nu \leq \delta/4$ and $\epsilon \leq \delta/(4|x^\nu|^2)$) to ensure that $\delta - \nu - \epsilon |x^\nu|^\nu$ is positive.

Since u and w_m are bounded (and thus trivially of subquadratic growth), it follows that Φ admits a maximum at $(t_\alpha, x_\alpha, s_\alpha, y_\alpha) \in ([0, T] \times \mathbb{R}^d)^2$ such that

$$\|u\|_\infty + \|w_m\|_\infty \geq u(t_\alpha, x_\alpha) - w_m(s_\alpha, y_\alpha) \geq \delta - \nu - \epsilon |x^\nu|^2 + \varphi(t_\alpha, x_\alpha, s_\alpha, y_\alpha). \quad (6.5)$$

Since $-\epsilon |x^\nu|^2 \geq -|x^\nu|^2$, the above inequality implies that

$$\alpha (|t_\alpha - s_\alpha|^2 + |x_\alpha - y_\alpha|^2) + \epsilon (|x_\alpha|^2 + |y_\alpha|^2)$$

is bounded independently of $\alpha > 0$ and $0 < \epsilon \leq 1$ (but not of ν since $|x^\nu|$ may be arbitrarily large).

Now, for fixed ϵ , consider some sequence of increasing α , say $(\alpha_n)_n$. To each α_n is associated a maximum point $(t_n, x_n, s_n, y_n) := (t_{\alpha_n}, x_{\alpha_n}, s_{\alpha_n}, y_{\alpha_n})$. By the discussion above, $\{(t_n, x_n, s_n, y_n)\}_n$ is contained in a compact set. Therefore, $(\alpha_n, t_n, x_n, s_n, y_n)_n$ admits a subsequence whose four last components converge to some point $(\hat{t}, \hat{x}, \hat{s}, \hat{y})$. With a slight abuse of notation, we relabel this subsequence $(\alpha_n, t_n, x_n, s_n, y_n)_n$, forgetting the original sequence. It follows that $\hat{x} = \hat{y}$ since otherwise $|\hat{x} - \hat{y}| > 0$ and Lemma 6.4 implies

$$\limsup_{n \rightarrow \infty} \{\alpha_n |x_n - y_n|^2\} = \limsup_{n \rightarrow \infty} \alpha_n |\hat{x} - \hat{y}|^2 = +\infty,$$

contradicting the boundedness in the discussion above. The same exact argument yields $\hat{t} = \hat{s}$. Moreover, letting $\varphi_n := \varphi(t_n, x_n, s_n, y_n; \alpha_n)$,

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow \infty} \varphi_n \leq \limsup_{n \rightarrow \infty} \{u(t_n, x_n) - w_m(s_n, y_n)\} - \delta + \nu + \epsilon |x^\nu|^2 \\ &\leq (u - w_m)(\hat{t}, \hat{x}) - \delta + \nu + \epsilon |x^\nu|^2 \end{aligned} \quad (6.6)$$

and hence

$$0 < \delta - \nu - \epsilon |x^\nu|^2 \leq (u - w_m)(\hat{t}, \hat{x}). \quad (6.7)$$

By Lemma 6.2, $(w_m - \mathcal{M}_\rho w_m)(s_n, y_n) \geq \xi/m$. Suppose, in order to arrive at a contradiction, $(\alpha_n, t_n, x_n, s_n, y_n)_n$ admits a subsequence along which $(u - \mathcal{M}_\rho u)(t_n, x_n) \leq 0$. As usual, we abuse slightly the notation and temporarily refer to this subsequence as $(\alpha_n, t_n, x_n, s_n, y_n)_n$. Combining these two inequalities,

$$\begin{aligned} -\xi/m &\geq u(t_n, x_n) - w_m(s_n, y_n) - (\mathcal{M}_\rho u(t_n, x_n) - \mathcal{M}_\rho w_m(s_n, y_n)) \\ &\geq \delta - \nu - \epsilon |x^\nu|^2 + \mathcal{M}_\rho w_m(s_n, y_n) - \mathcal{M}_\rho u(t_n, x_n). \end{aligned}$$

Taking limit inferiors with respect to $n \rightarrow \infty$ of both sides of this inequality and using the semicontinuity established in Lemma 5.1 yields

$$-\xi/m \geq \delta - \nu - \epsilon |x^\nu|^2 + \mathcal{M}_\rho w_m(\hat{t}, \hat{x}) - \mathcal{M}_\rho u(\hat{t}, \hat{x}).$$

It follows, by the upper semicontinuity of u , that the supremum in $\mathcal{M}_\rho u(\hat{t}, \hat{y})$ is achieved at some $\hat{z} \in Z$. Therefore,

$$-\xi/m \geq \delta - \nu - \epsilon|x^\nu|^2 + w_m(\hat{t}, \Gamma(\hat{t}, \hat{x}, \hat{z})) - u(\hat{t}, \Gamma(\hat{t}, \hat{x}, \hat{z})) \geq -\nu - \epsilon|x^\nu|^2.$$

Taking ν and ϵ small enough yields a contradiction. By virtue of the above, we may assume that our original sequence $(\alpha_n, t_n, x_n, s_n, y_n)_n$ whose three last components converge to $(\hat{t}, \hat{x}, \hat{s}, \hat{y})$ satisfies $(u - \mathcal{M}_\rho u)(t_n, x_n) > 0$ for all n .

Now, suppose $\hat{t} = T$. By Lemma 6.2, $(w_m - \mathcal{M}_\rho w_m)(T, \hat{x}) \geq \xi/m$ and $w_m(T, \hat{x}) - g(\hat{x}) \geq 0$. If $(u - \mathcal{M}_\rho u)(T, \hat{x}) \leq 0$, we arrive at a contradiction by an argument similar to the above. It follows that $u(T, \hat{x}) - g(\hat{x}) \leq 0$ and hence $(u - w_m)(T, \hat{x}) \leq 0$, contradicting (6.7). We conclude that $\hat{t} < T$ so that we may safely assume $(t_n, x_n, s_n, y_n) \in \mathcal{O}$ for all n .

We are now in a position to apply the Crandall-Ishii lemma [15, Theorem 3.2], which implies the existence of $X_n, Y_n \in \mathcal{S}(d)$ satisfying²

$$\begin{aligned} (\partial_t \varphi_n, D_x \varphi_n, X_n + 2\epsilon I_d) &\in \text{cl}(\mathcal{P}_{\mathcal{O}}^{2,+} u)(t_n, x_n), \\ (-\partial_s \varphi_n, -D_y \varphi_n, Y_n - 2\epsilon I_d) &\in \text{cl}(\mathcal{P}_{\mathcal{O}}^{2,-} w_m)(s_n, y_n), \end{aligned}$$

and

$$\begin{aligned} -(3\alpha_n + \epsilon) I_{2d} &\preceq \begin{pmatrix} X_n & \\ & -Y_n \end{pmatrix} + 2\epsilon I_{2d} \\ &\preceq (3\alpha_n + 2\epsilon) \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} + (\epsilon + \epsilon^2/\alpha) I_{2d}. \end{aligned}$$

Due to our choice of φ , we get

$$\begin{aligned} a_n := \partial_t \varphi_n &= \partial_t \varphi(t_n, x_n, s_n, y_n; \alpha_n) = \alpha_n (t_n - s_n) \\ &= -\partial_s \varphi(t_n, x_n, s_n, y_n; \alpha_n) = -\partial_s \varphi_n \end{aligned}$$

along with

$$D_x \varphi_n = \alpha_n(x_n - y_n) + \epsilon x_n \text{ and } D_y \varphi_n = \alpha_n(x_n - y_n) - \epsilon y_n.$$

Therefore, since $(u - \mathcal{M}_\rho u)(t_n, x_n) > 0$,

$$\begin{aligned} -a_n + H(t_n, x_n, u(t_n, x_n), \alpha_n(x_n - y_n) + \epsilon x_n, X_n + 2\epsilon I_d) &\leq 0 \\ \text{and } -a_n + H(s_n, y_n, w_m(s_n, y_n), \alpha_n(x_n - y_n) - \epsilon y_n, Y_n - 2\epsilon I_d) &\geq 0. \end{aligned} \quad (6.8)$$

Moreover, if α_n is large enough (e.g., $\alpha_n \geq 3$), the chain of inequalities involving X_n and Y_n imply

$$-4\alpha_n I_{2d} \preceq \begin{pmatrix} X_n & \\ & -Y_n \end{pmatrix} \preceq 4\alpha_n \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix},$$

and we can combine the inequalities (6.8) and apply Lemma 6.3 to get

$$\begin{aligned} 0 &\leq H(s_n, y_n, w_m(s_n, y_n), \alpha_n(x_n - y_n) - \epsilon y_n, Y_n - 2\epsilon I_d) \\ &\quad - H(t_n, x_n, u(t_n, x_n), \alpha_n(x_n - y_n) + \epsilon x_n, X_n + 2\epsilon I_d) \\ &\leq \rho(w_m(s_n, y_n) - u(t_n, x_n)) + \text{const.}(\varphi_n + \epsilon) + \omega(|(t_n, x_n) - (s_n, y_n)|) \end{aligned} \quad (6.9)$$

²The elliptic Crandall-Ishii actually gives us $(X_n, Y_n) \in \mathcal{S}(d+1)$. An argument using the fact that the principal submatrices of a positive semidefinite (PSD) matrix are PSD allows us to discard the extra dimension associated with time.

where ω is a modulus of continuity. Moreover, by (6.5),

$$w_m(s_n, y_n) - u(t_n, x_n) \leq -\delta + \nu + \epsilon |x^\nu|^2, \quad (6.10)$$

and by (6.6),

$$\limsup_{n \rightarrow \infty} \varphi_n \leq \nu + \epsilon |x^\nu|^2. \quad (6.11)$$

Applying (6.10) to (6.9), taking the limit superior as $n \rightarrow \infty$ of both sides, and finally applying (6.11) to the resulting expression yields

$$\delta \leq \text{const.} (\nu + \epsilon + \epsilon |x^\nu|^2)$$

(note that const. above is not the same as in (6.9) and may depend on ρ). Picking ν small enough and taking $\epsilon \rightarrow 0$ yields the desired contradiction. \blacksquare

7 Value of the game

We prove, in this section, Theorem 2.13. The advantage of using strategies with delay is access to the following result.

Lemma 7.1. *For each $(\alpha, \beta) \in \mathcal{A}(t) \times \mathcal{B}(t)$, there exists $(a, b) \in \mathcal{A}(t) \times \mathcal{B}(t)$ such that $\alpha(b) = a$ and $\beta(a) = b$.*

A proof of the above is given in [7, Lemma 2.4]. In particular, Lemma 7.1 allows us to unambiguously define $J(t, x; \alpha, \beta) := J(t, x; a, b)$ (where (a, b) are associated to (α, β) as in the lemma statement).

Proof of Theorem 2.13. Let $(t, x) \in \text{cl } \mathcal{O}$. By Lemma 7.1,

$$v^-(t, x) = \sup_{\alpha \in \mathcal{A}(t, x)} \inf_{\beta \in \mathcal{B}(t)} J(t, x; \alpha, \beta) \leq \inf_{\beta \in \mathcal{B}(t)} \sup_{\alpha \in \mathcal{A}(t, x)} J(t, x; \alpha, \beta) = v^+(t, x).$$

The reverse inequality is an immediate consequence of Theorems 2.11 and 2.12. \blacksquare

Remark. Previous works in impulse control games establish $v^+ = v^-$ by deriving stronger DPPs heuristically (see [36, Lemma 3.4] and [13, Theorem 4.1]). However, these DPPs cannot be obtained rigorously. In fact, it is this very issue that led Fleming and Souganidis to employ the method of π -strategies [19, Section 2] in their pioneering work on SDGs. The delay framework serves as a simple alternative, and can, with little hassle, be used to strengthen the results of [36, 13].

Remark. Other recent arguments/frameworks used to establish that an SDG admits a value are path-dependent PDEs [29] and the direction of Mihai Sirbu and his collaborators [32, 3], who employ the stochastic Perron's method on games, allowing for a symmetric formulation in which both players play controls.

8 Extensions

We mention that by adapting the technique in [31, Theorem 5.11], one should be able to extend the comparison principle to solutions of arbitrary polynomial growth. However, for polynomial degree d growth, the resulting uniqueness theorem requires the existence of a

classical “strict” supersolution $c := c(t, x)$ (similar to Lemma 6.2) satisfying $c(t, x)/|x|^p \rightarrow \infty$ as $|x| \rightarrow \infty$. Unfortunately, the construction of such a solution is ad hoc (i.e., problem dependent); see the discussion at the end of [31, Section 2.5].

In the interest of generality, we point out that Assumptions 2.7 (ii) and 2.8 are not used in the proof of comparison. We also mention a few trivial extensions: removing altogether the requirement $K(t, z) \in \omega(1)$ of Assumption 2.7 (iii) and redefining $Z := Z(t, x)$, $\Gamma := \Gamma(t, x, z)$, and $K := K(t, x, z)$ to depend on x with

$$\mathcal{M}u(t, x) := \sup_{z \in Z(t, x)} \{u(t, x + \Gamma(t, x, z)) + K(t, x, z)\}$$

does not invalidate the comparison principle if we require that Z is pointwise compact and $Z(t_n, x_n) \rightarrow Z(t, x)$ in the Hausdorff metric whenever $(t_n, x_n) \rightarrow (t, x)$ in order for Lemmas 5.1 and 6.1 to remain valid (see [27, Lemma 5.1] and [31, Lemma 4.3]). Moreover, in the extension above, we can additionally exchange the condition $K \leq -K_0$ pointwise for $K \geq K_0$ pointwise (corresponding to a reward per impulse). In this case, one uses the family of “strict” subsolutions to (6.1) defined by $\{u_\lambda := (1 + \lambda)u - \lambda c\}_{\lambda \in (0, 1)}$ instead of $\{w_\lambda\}_\lambda$ in the proof of comparison (the argument to show that u_λ is a subsolution is similar to the proof of Lemma 6.2, using the anti-convexity of \mathcal{M} instead of its convexity). We mention that this is just one possible way to “tweak” the conditions in such a way that does not invalidate comparison.

The setting of the above paragraph is important namely because it appears in practical impulse control problems (e.g., optimal consumption with fixed and proportional transaction costs [1, Section 6.2] and guaranteed minimum withdrawal benefits in variable annuities [1, Section 6.3]). However, the reader will find (with some reflection) that defining v^\pm when Γ , K , and Z depend on x is a nontrivial matter.

Acknowledgements. The author wishes to express his greatest gratitude to Catherine Rainer (Université de Brest) for all of her help, support, and for pointing out the use of delay strategies to establish the pointwise inequality $v^- \leq v^+$.

The author also thanks Andrea Cosso (Politecnico di Milano) for discussion regarding DPPs, Christine Grün (Université Toulouse 1 Capitole) for discussion on games, and Nizar Touzi (École Polytechnique) for discussion regarding comparison.

A Equivalent definition of viscosity solutions

Since the DPP (Theorem 2.9) holds only for stopping times taking countably many values, we are unable to use début-type stopping times in order to derive the DPE (Theorem 2.11). In this case, to apply Dynkin’s formula, we require our test functions to be compactly supported. The following result affords us this luxury:

Lemma A.1. *Let Definition 2.10_c refer to Definition 2.10 with $C^{1,2}(\mathcal{O})$ replaced by $C_c^{1,2}(\mathcal{O})$. A subsolution (resp. supersolution) under Definition 2.10 is a subsolution (resp. supersolution) under Definition 2.10_c and vice versa.*

Proof. One direction is trivial, since $C_c^{1,2}(\mathcal{O}) \subset C^{1,2}(\mathcal{O})$.

Suppose u is a subsolution under Definition 2.10_c and let $(t, x, \varphi) \in \mathcal{O} \times C^{1,2}(\mathcal{O})$ be given as in Definition 2.10 (we need not consider the parabolic boundary $\partial^+ \mathcal{O}$, as it is not “tested”

by φ). For brevity, let $B_r := B(x; r)$. Let $\psi \in C_c^{1,2}(\mathcal{O})$ be given by

$$\begin{aligned}\psi(t, x) &:= \varphi(t, x) \mathbf{1}_{B_1}(x) + \zeta(x) \varphi(t, \hat{x}) \mathbf{1}_{B_2 \setminus B_1}(x) \\ \text{where } \zeta(x) &:= \exp(1 - 1/(1 - |x - \hat{x}|^4))\end{aligned}$$

and $\hat{x} \in \text{cl } B_1$ is the (unique) closest point to x in $\text{cl } B_1$. Intuitively, ζ is used to mollify the value of ψ on ∂B_1 (where it is equal to φ) with its value on ∂B_2 (where it is equal to zero). Note that ψ inherits all the local properties of φ at (t, x) since the two functions coincide on an open ball. Therefore,

$$F(t, x, u^*, D\varphi(t, x), D^2\varphi(t, x)) = F(t, x, u^*, D\psi(t, x), D^2\psi(t, x)) \leq 0 \text{ (resp. } \geq 0),$$

as desired. The supersolution case is identical. \blacksquare

B Existence by Perron's method

This appendix gives an alternative existence proof for the HJBI (2.3) using only the fact that v^- is itself a supersolution and the following assumption:

Assumption B.1. for each sequence $(t_n, x_n)_n$ taking values in $[0, T] \times \mathbb{R}^d$ and converging to some (T, x) , $v^-(t_n, x_n) \rightarrow g(x)$.

We first analyze the simpler Hamilton-Jacobi-Bellman (HJB) equation

$$0 = G(\cdot, u(\cdot), Du(\cdot), D^2u(\cdot)) := \begin{cases} -\partial_t u + H & \text{on } \mathcal{O} \\ u - g & \text{on } \partial^+ \mathcal{O} \end{cases} \quad (\text{B.1})$$

where H is defined in (6.3). We take ρ appearing in H to be zero. This choice is purely to simplify the treatment, as a trivial change of variables establishes our results for the case of positive ρ . We thus write $H := H(t, x, q, X)$ throughout this section.

Below, we relate the HJB to the “single-player” value function

$$v(t, x) := \inf_{b \in \mathcal{B}(t)} J(t, x; \hat{a}, b)$$

where $\hat{a} \in \mathcal{A}(t)$ denotes a control without impulses. Similarly to Theorem 2.9, we can show that v satisfies a DPP:

Lemma B.2. For each $(t, x) \in \mathcal{O}$ and each $[t, T]$ -valued family of $(\mathcal{F}_{t,s})_{s \in [t, T]}$ -stopping times $\{\theta^b\}_{b \in \mathcal{B}(t)}$,

$$\begin{aligned}v(t, x) &\leq \inf_{b \in \mathcal{B}(t)} \mathbb{E} \left[\int_t^\theta f(s, X_s, b_s) ds + v^*(\theta, X_\theta) \right] \\ \text{and } v(t, x) &\geq \inf_{b \in \mathcal{B}(t)} \mathbb{E} \left[\int_t^\theta f(s, X_s, b_s) ds + v_*(\theta, X_\theta) \right] \end{aligned} \quad (\text{B.2})$$

where it is understood that $X := X^{t, x; \hat{a}, b}$ and $\theta := \theta^b$.

Using the above DPP, we can show that v is both a subsolution and supersolution of (B.1). The proof is similar to that of Theorem 2.11, made even simpler by the fact that the DPP holds for stopping times taking values in $[t, T]$ (not just \mathbb{Q}_t).

Lemma B.3. *v is a bounded solution of (B.1).*

A comparison principle for the HJB can be established similarly to Theorem 2.12. The proof is made even simpler by the lack of obstacle.

Lemma B.4. *If u is a bounded subsolution and w is a bounded supersolution of (B.1), $u^* \leq w_*$ pointwise.*

The following is a direct consequence of the above lemmas.

Corollary B.5. *v is a continuous and bounded solution to (B.1), unique among all bounded solutions. Moreover, v is a subsolution of (2.3).*

Another implication (though we do not use it) is that the DPP (B.2) can be strengthened (a posteriori) to hold with equality since $v_* = v = v^*$.

Lemma B.6. *Let Y be a topological space, $B \subset \mathbb{R}^{d_B}$ be compact, $h: Y \times B \rightarrow \mathbb{R}$ be continuous, and $(y_n)_n$ be a sequence in Y such that $y_n \rightarrow \hat{y} \in Y$. For each n , pick $b_n \in B$ such that $h(y_n, b_n) = \sup_{b \in B} h(y_n, b)$. Then, for any cluster point \hat{b} of $(b_n)_n$, $h(\hat{y}, \hat{b}) = \sup_{b \in B} h(\hat{y}, b)$.*

Proof. The claim $h(\hat{y}, \hat{b}) \leq \sup_{b \in B} h(\hat{y}, b)$ is trivial, so we need only show the reverse inequality. With a slight abuse of notation, we denote by $(y_n, b_n)_n$ a subsequence converging to (\hat{y}, \hat{b}) . By continuity and the definition of b_n ,

$$h(\hat{y}, \hat{b}) = \lim_{n \rightarrow \infty} h(y_n, b_n) = \lim_{n \rightarrow \infty} \sup_{b \in B} h(y_n, b).$$

Moreover, for each $\epsilon > 0$, there exists a $b^\epsilon \in B$ such that

$$\sup_{b \in B} h(\hat{y}, b) \leq h(\hat{y}, b^\epsilon) + \epsilon = \lim_{n \rightarrow \infty} h(y_n, b^\epsilon) + \epsilon \leq \lim_{n \rightarrow \infty} \sup_{b \in B} h(y_n, b) + \epsilon.$$

Combining these two relations and taking ϵ to zero yields the desired result. ■

Lemma B.7. *Let $Y \subset \mathbb{R}^n$ be closed, $B \subset \mathbb{R}^{d_B}$ be compact, and $h: Y \times B \rightarrow \mathbb{R}$ be continuous. Then, $y \mapsto \sup_{b \in B} h(y, b)$ is continuous.*

Proof. Let $y \in Y$, $\epsilon > 0$, and $C(y)$ denote the closed unit ball in \mathbb{R}^n centred at y . Since h restricted to $(Y \cap C(y)) \times B$ is uniformly continuous, we can pick $0 < \delta \leq 1$ such that for all $y' \in Y$ with $|y - y'| < \delta$, $|h(y, b) - h(y', b)| < \epsilon$. ■

We are now in a position to employ Perron's method. The main idea exploits the following relationship between HJBI (2.3) and HJB (B.1):

$$\begin{aligned} u \text{ is a subsolution of (B.1)} &\implies u \text{ is a subsolution of (2.3);} \\ u \text{ is a supersolution of (2.3)} &\implies u \text{ is a supersolution of (B.1).} \end{aligned}$$

Lemma B.8. *Let \mathcal{W} be a nonempty family of bounded supersolutions to (B.1) and define $u: \text{cl } \mathcal{O} \rightarrow \mathbb{R}$ by $u(t, x) := \inf_{w \in \mathcal{W}} \{w(t, x)\}$. Then, $u \in \mathcal{W}$.*

Proof. Corollary B.5 provides a bounded and continuous subsolution v of (B.1). By Lemma B.4, $v \leq w_*$ holds on $\text{cl } \mathcal{O}$ for each $w \in \mathcal{W}$ and hence

$$v(t, x) \leq \inf_{w \in \mathcal{W}} \{w_*(t, x)\} \leq \inf_{w \in \mathcal{W}} \{w(t, x)\} = u(t, x) \text{ for } (t, x) \in \text{cl } \mathcal{O}.$$

Since v is continuous, $g = v \leq u_*$ on $\partial^+ \mathcal{O}$.

Now, without loss of generality, let $(t, x, \varphi) \in \mathcal{O} \times C^{1,2}(\mathcal{O})$ be such that $(u_* - \varphi)(t, x) = 0$ is a local minimum of $u_* - \varphi$. We may assume that for some $h > 0$ such that $t + h < T$,

$$(u_* - \varphi)(s, y) \geq |t - s|^2 + |x - y|^4 \text{ for } (s, y) \in B(t, x; h).$$

Let $(t_n, x_n)_n$ be a sequence such that $(t_n, x_n) \rightarrow (t, x)$, $(t_n, x_n) \in B(t, x; h)$, and $u(t_n, x_n) \rightarrow u_*(t, x)$. The definition of u also affords us a sequence $(u_n)_n$ of supersolutions in \mathcal{W} such that

$$(u - \varphi)(t_n, x_n) \geq ((u_n)_* - \varphi)(t_n, x_n) - 1/n.$$

For each n , pick $(s_n, y_n) \in \text{cl}(B(t, x; h) \cap \mathcal{O})$ such that

$$((u_n)_* - \varphi)(s_n, y_n) = \min_{(s, y) \in \text{cl}(B(t, x; h) \cap \mathcal{O})} ((u_n)_* - \varphi)(s, y).$$

It follows that

$$\begin{aligned} (u - \varphi)(t_n, x_n) + 1/n &\geq ((u_n)_* - \varphi)(t_n, x_n) \geq ((u_n)_* - \varphi)(s_n, y_n) \geq (u - \varphi)(s_n, y_n) \\ &\geq (u_* - \varphi)(s_n, y_n) \geq |t - s_n|^2 + |x - y_n|^4. \end{aligned}$$

With a slight abuse of notation, we pick a subsequence of $(s_n, y_n)_n$ converging to (\hat{s}, \hat{y}) and relabel it $(s_n, y_n)_n$. Taking limits in the above inequality, we see that $(\hat{s}, \hat{y}) = (t, x)$. This implies that for n large enough, $((u_n)_* - \varphi)(s_n, y_n)$ is a local minimum of $(u_n)_* - \varphi$. Since u_n is a viscosity supersolution,

$$-\partial_t \varphi(s_n, y_n) + H(s_n, y_n, D_x \varphi(s_n, y_n), D_x^2 \varphi(s_n, y_n)) \geq 0.$$

By an argument involving Lemma B.6, we can take limits (along an appropriate subsequence) to yield

$$-\partial_t \varphi(t, x) + H(t, x, D_x \varphi(t, x), D_x^2 \varphi(t, x)) \geq 0,$$

concluding $u \in \mathcal{W}$. ■

Lemma B.9. *Let \mathcal{W} be a nonempty family of bounded supersolutions to (2.3) and define $u : \text{cl } \mathcal{O} \rightarrow \mathbb{R}$ by $u(t, x) := \inf_{w \in \mathcal{W}} \{w(t, x)\}$. Then, $u \in \mathcal{W}$.*

Proof. Since $u_* \leq u \leq w_*$ for each $w \in \mathcal{W}$, Lemma 5.1 implies

$$0 \leq w_* - \mathcal{M}w_* \leq w - \mathcal{M}u_* \text{ on } \text{cl } \mathcal{O} \text{ for each } w \in \mathcal{W}.$$

Therefore,

$$(u - \mathcal{M}u_*)(t, x) = \inf_{w \in \mathcal{W}} \{(w - \mathcal{M}u_*)(t, x)\} \geq 0 \text{ for } (t, x) \in \text{cl } \mathcal{O}.$$

Rearranging and taking lower semicontinuous envelopes yields $u_* \geq \mathcal{M}u_*$ on $\text{cl } \mathcal{O}$ by an application of Lemma 5.1.

The desired result follows by noting that any supersolution of (2.3) is a supersolution of (B.1) and applying Lemma B.8. ■

Lemma B.10. *Let \mathcal{W} be the set of bounded supersolutions to (2.3). If $w \in \mathcal{W}$ is not a subsolution of (2.3), there exists $\tilde{w} \in \mathcal{W}$ such that $w > \tilde{w}$ at a point in $\text{cl } \mathcal{O}$.*

Proof. If $w \in \mathcal{W}$ is not a subsolution of (2.3), there either exists $(t, x, \varphi) \in \mathcal{O} \times C^{1,2}(\mathcal{O})$ such that $(w^* - \varphi)(t, x)$ is a local maximum of $w^* - \varphi$ and

$$\min \{ -\partial_t \varphi(t, x) + H(t, x, D_x \varphi(t, x), D_x^2 \varphi(t, x)), (w^* - \mathcal{M}w^*)(t, x) \} > 0, \quad (\text{B.3})$$

or an $x \in \mathbb{R}^d$ such that

$$\min \{ w^*(T, x) - g(x), (w^* - \mathcal{M}w^*)(T, x) \} > 0. \quad (\text{B.4})$$

First, consider the case of (B.3). Without loss of generality, we assume w is lower semi-continuous. Noting that H is continuous by Lemma B.7, we employ a “bump” construction as in [15, pg. 24] to get a constant h and a smooth function $\psi: \text{cl } \mathcal{O} \rightarrow \mathbb{R}$ such that $h > 0$, $t + h < T$, ψ is a solution of

$$\min \{ -\partial_t u(\cdot) + H(\cdot, D_x u(\cdot), D_x^2 u(\cdot)), (u - \mathcal{M}w^*)(\cdot) \} \geq 0 \text{ on } B(t, x; h) \cap \mathcal{O},$$

with $w < \psi$ on the annulus $\text{cl}(B(t, x; h)) \setminus B(t, x; h/2)$, and

$$\tilde{w} := \begin{cases} \min \{ w, \psi \} & \text{on } B(t, x; h) \\ w & \text{otherwise} \end{cases}$$

is dominated by w at a point. Applying Lemma B.8, we get that \tilde{w} is a supersolution of (B.1) satisfying $\tilde{w}_* - \mathcal{M}w \geq 0$. Moreover, since $\tilde{w}_* \leq \tilde{w} \leq w$ on $\text{cl } \mathcal{O}$, we conclude $\tilde{w}_* - \mathcal{M}\tilde{w}_* \geq 0$ on $\text{cl } \mathcal{O}$, so that \tilde{w} is indeed a supersolution.

In the case of (B.4), Assumption B.1 yields

$$(w^* - v^-)(T, x) > \max \{ 0, \mathcal{M}w^*(T, x) - g(x) \} \geq 0.$$

Letting $(t_n, x_n)_n$ denote a sequence such that $(t_n, x_n) \rightarrow (T, x)$ and $w(t_n, x_n) \rightarrow w^*(T, x)$, we have that $\lim_{n \rightarrow \infty} (w - v^-)(t_n, x_n) > 0$. Particularly, this implies the existence of an n such that $w(t_n, x_n) > v^-(t_n, x_n)$. Since v^- is a bounded supersolution of (2.3) by Theorem 2.11, we conclude by taking $\tilde{w} := v^-$. ■

Theorem B.11. *There exists a bounded and continuous solution of (2.3), unique among all bounded solutions.*

Proof. Let \mathcal{W} be given as in Lemma B.10 and $u(t, x) := \inf_{w \in \mathcal{W}} \{ w(t, x) \}$. By Lemma B.9, u is a supersolution of (2.3). Moreover, if u is not a subsolution of (2.3), Lemma B.10 guarantees the existence of a supersolution smaller than u at a point, contradicting the pointwise minimality of u among supersolutions. ■

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